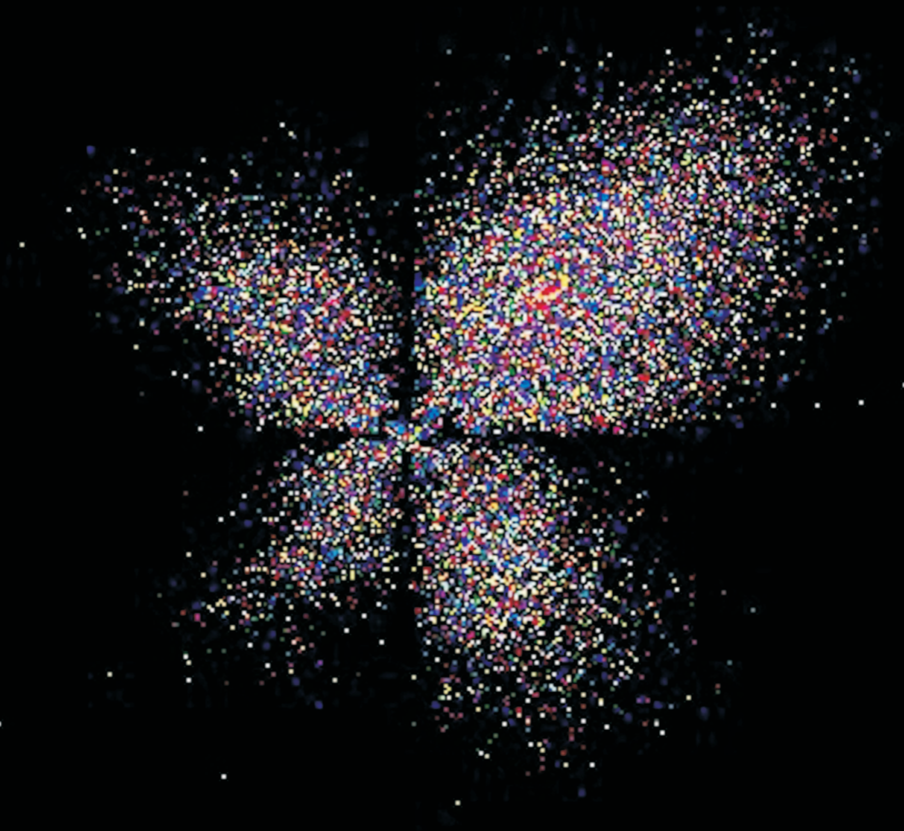
Wave-Particles

Suggestions on Field Unification, Dark Matter and Dark Energy

Alberto Strumia



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Preface

The history of modern physics taught us that several phenomena which, at first, appeared to their observers as independent and unrelated with each other, later on have been recognized as different aspects of a same physical reality. The theories which brought to conceptual unification of such apparently unrelated aspects of the physical world, generally took their first steps when it was realized that, even if different, such phenomena presented systematically some common occurrences.

1. Sometimes the way leading to unification has been suggested just by similar, but regular coincidences, as it was the case of *general relativity,* born by the recognition of coincidence between *inertial* and *gravitational* masses, the values of which are always directly proportional.

   What does it happen if they are taken as being the "same thing", observed in different conditions? The circumstance that the *gravitational forces* and the *apparent forces* arising respect to a non inertial frame are locally indistinguishable, assumed as a new physical principle (today known as *equivalence principle*) allowed Albert Einstein (1879-1955) to develop his theory of *general relativity* (1916).

2. Previously, the unification between *electricity* and *magnetism* was suggested, to James Clerk Maxwell (1831-1879), by symmetry criteria concerning the mathematical laws governing electrodynamics. In fact the equations of electromagnetic fields become more symmetric if one introduces the *displacement current* term.

   It was starting from such assumption that the electromagnetic waves were predicted. Later, *special relativity* has shown as electric and magnetic fields are the components of the same antisymmetric tensor field $\mathsf{F} \equiv (F_{\mu\nu})$.

3. Moreover a partial unification between *waves* and *particles* was performed by *quantum mechanics.*

   Einstein's interpretation of photoelectric effect (1905), for which he was awarded the Nobel prize (1921), showed that light, which until that time was treated as a *wave* (after the failure of Newton's corpuscular theory), at least in the case of the photoelectric effect, behaves as a flow of discrete *particles* (*photons*), the energy of each one being proportional to the wave frequency according to the equation:

$$E = h\nu,$$

which became famous as *Einstein-Planck relation,* h being the now well known Planck constant.

A further step towards wave-particle unification was made when Louis de Broglie (1892-1987) proposed his *matter waves,* associated to particle motion according to the equation, later called *De Broglie relation,* between the wave length λ and the momentum p of the particle:

$$\lambda = \frac{h}{p}.$$

Following a more advanced mathematical approach, which actually gave rise to *quantum mechanics,* Erwin Schrödinger (1887-1961) pointed out that the variational equation of light rays in the context of *geometrical optics* (which can be obtained as approximation of *wave optics* when the wave lengths are negligible respect to the optical paths):

$$\delta \int \boldsymbol{k} \cdot \mathrm{d}\boldsymbol{x} = 0,$$

and the variational equation of particle trajectories, provided by *analytical mechanics*:

$$\delta \int \boldsymbol{p} \cdot \mathrm{d}\boldsymbol{x} = 0,$$

were formally the same and could be identified assuming that:

$$\boldsymbol{p} = \hbar\,\boldsymbol{k}.$$

In fact, if one evaluates the modulus in the last equation, one just obtains De Broglie relation, since $\hbar = h/2\pi$ is the reduced Planck constant and $k = 2\pi/\lambda$ is the wave number.

In the frame of Newtonian (*i.e.,* non relativistic) *analytical mechanics* of conservative systems, the variational principle can be written also in the equivalent form:

$$\delta \int n\,\mathrm{d}s = 0,$$

which represents, in a variational formulation, the *Fermat's principle* of shorter optical path (or least time), n being the refractive index of the medium across which light propagates. And:

$$\delta\int\sqrt{2m(E-V)}\,\mathrm{d}s=0,$$

is the *Maurpertuis' principle* for the determination of particle trajectory of motion, δ being here the *isoenergetic variational operator*.

After such a comparison the idea that *analytical mechanics* was an approximation of an exact still unknown *wave mechanics,* like *geometrical optics* is an approximation of *wave optics,* was seriously considered. The equation candidate to govern such wave mechanics was obtained by Schrödinger, through a backward way, leading to his famous equation:

$$i\hbar\frac{\partial\psi}{\partial t}=\boldsymbol{H}\psi,$$

where here the Hamiltonian $\boldsymbol{H}$ has become a differential operator.

At the present stage of unification process we can wonder wether a further step is legitimate in order to fully unify waves and particle approaches.

Can waves and particles be viewed as two different ways of interpreting the same equation of motion arising from a continuos unified field of some nature?

Were it possible, the wave-particle unification would be actually complete.

A single equation could be viewed, at the same time, as *equation of motion of a wave-front* and as *equation of motion of a particle,* or better of a *family of particles.* And the theory would no longer be approximated, but exact. Since it would no longer identify only *optical rays* with mechanical *particle trajectories,* but also the *evolution laws* of their motions along the respective paths. In other words the identification would involve the *space-time paths.*

A wave-particle family gathers together all the *initial conditions* which determine all the *actual states* or wave functions at some time and place, collected within an infinite component vector in some Hilbert space. At a macroscopic (classical) level the observer himself chooses the initial conditions, so identifying an individual particle. While at a microscopical (*quantum*) level the observer can only select some interval where the initial conditions live, because of the *uncertainty principle* which arises because of the non-commutative algebra of the operators. At an intermediate (*semiclassical*) level the observer is able to approximate the uncertainty volume in phase space by choosing a point representative of the volume itself, so interpreting the probability cloud as a single particle.

Starting from similar and other questions we have attempted to attack the problem of such unification in the present volume.

The book is organized in two parts, as follows:

1. In the first part:
   (a) *Chapter 1* introduces the problem following a heuristic approach which suggests a meaningful and intriguing way to attack the matter.
   (b) *Chapter 2* offers a *non-explicitly covariant formulation* of the proposed wave-particle unified mechanics, referred to a synchronous frame (*i.e.,* under an assumption which simplifies the formalism), while
   (c) *Chapter 3* extends the same results to an *explicitly covariant formulation* of the theory to a quite general frame, with no co-ordinate choice assumptions.
2. In the second part,

   which is devoted to investigate possible models for the unification of physical (interaction and matter) fields the solutions of which are interpreted as wave-particles, according to the approach presented in the first part of the volume,
   (a) *Chapter 4* presents an attempt to test usual Kaluza-Klein theories as possible candidates. It is shown that such an approach is unsatisfactory.
   (b) *Chapter 5* proposes a new original perspective according to which the fields governing the fundamental interactions and matter are included within the eigenvectors of the metric tensor of a suitable space-time endowed with more than four dimensions.
   (c) *Chapter 6* is dedicated to a physical interpretation of the theory in order to fit the fundamental interaction fields (*bosons*) of the *standard model* of elementary particles theory.

(d) *Chapter 7* is devoted to a physical interpretation of the theory in order to fit the matter fields (*fermions*) of the *standard model* of elementary particles theory.

(e) *Chapter 8* shows possible applications to cosmology including suggestions to interpret new energy-momentum tensor contributions as related to dark matter and dark energy.

(f) *Chapter 9* deals with a possible way to field quantization including quantum gravity.

Some concluding remarks and a bibliography end the whole volume.

Contents

Part I

Wave-Particles

The *Part I* of the book deals with unification of wave and particle equations of motion.

The same partial differential equation is interpreted either as wave-front equation of motion or as Hamilton-Jacobi equation governing the dynamics of a family of particles.

Chapter 1

Heuristic Approach

Abstract
Conditions required in order to identify a wave-front propagation *p.d.e* and a Hamilton-Jacobi equation governing motion of a family of particles, are presented following a heuristic approach. We show how such identification, which unifies mathematically waves and particles can be reached if and only if both waves and particles travel at the speed of light so that particles result to be massless. A rest mass may be allowed in the observable four-dimensional space-time if this latter is embedded within a higher dimensional one. Interaction potentials are replaced by a suitable metric and connection.

1.1 The Historical Context

It is known that Louis De Broglie, at the very beginning of quantum mechanics – attempting to reconcile wave-particle dual behavior of matter – conjectured that the particle motion was guided by a sort of *pilot wave*. (His original papers are collected in [13]. An historical survey on the relationship between relativity and quantum mechanics is presented, *e.g.,* in [18]).

Later David Bohm proposed the so called *realistic* interpretation of Schrödinger equation (see [4]). In both cases something artificial was required to provide consistency to the theory, like the request that the particle velocity is different from the wave *phase speed,* rather being equal to the wave *group velocity,* so that particles generally travel slowly than waves. As a consequence we cannot speak, properly of a conceptual unification between waves and particles, but rather of *complementarity* of two ways of representing some physical reality, as it was suggested by Niels Bohr (A wide survey on Bohr "philosophy" about *complementarity* may be found in [5]). A solution that may appear somehow unsatisfactory. A further step, in order to identify even the wave and particle speeds, was made later by Guy Boillat who advanced the idea – which was especially appreciated by Leopold Infeld – that *stable particles* could be thought as *exceptional discontinuity waves* traveling across the physical space and running along a path which represents, at the same time, a particle *trajectory* and a wave *ray.* His approach is significantly different respect to the previous ones since he operates in the context of a quite general *non-linear* wave propagation theory obtaining also the well known Born-Infeld electrodynamics and some other relevant results. (The most relevant results about the theory are collected in [6,8] and [9]).

Here we intend to investigate a deeper level of unification, by requiring that beside the *geometry* (*i.e.,* particle trajectory and wave rays in physical 3-*dimensional* space) also the *kinematics* (*i.e.,* the evolution laws of motion) are the same. Therefore we will impose that both the *Hamilton-Jacobi evolution law* along wave rays and the *Hamilton-Jacobi equation of motion* governing particles motion are the same equation. In other words we suggest to identify the world lines describing waves and particles in a relativistic space-time, so probably opening also a way towards relativity and quantum mechanics unification. As a consequence it will result that waves and particles run on the same ray/trajectory and travel at the same velocity, which is equal to the wave *phase velocity.*

After the present introductory section, the chapter approaches the problem in a non-explicitly covariant formulation, beginning, heuristically, to work within a 3-*dimensional* physical space. The problem itself will lead us naturally into a relativistic context.

In the next chapters 2 and 3, the same results will be represented with more rigor and generality.

1.2 Identifying Wave and Particle Mechanics

In the present section we will proceed following three subsequent steps: in the first step we show how to identify the wave-front motion equation and the Hamilton-Jacobi equation; in the second one we introduce a space metric in order to describe interaction forces and in the third step we show how to provide particle rest mass embedding the observable 4-*dimensional* space-time within a higher dimensional one.

1.2.1 First Step – Wave-Front and Particle Motion

Let us start, heuristically, considering the 3-*dimensional* physical space. The co-ordinates x^i of each point $\boldsymbol{x}$ are labelled, as usual, by the Latin indices $i = 1, 2, 3$. We now consider any scalar differentiable real valued function $\varphi(t, \boldsymbol{x})$, which we can always assume to be dimensionless. The equation:

$$\varphi(t, \boldsymbol{x}) = 0, \tag{1.1}$$

may be interpreted as the equation of motion of a wave-front traveling across the physical space. We point out that φ is determined by (1.1) except for an arbitrary non-vanishing factor. A degree of freedom which will be important later in order to guarantee the constance of particle mass (see §1.2.3).

The trajectory (*ray*) of each point of the wave-front can be described by its parametric equation:

$$\boldsymbol{x} \equiv \boldsymbol{x}(t). \tag{1.2}$$

Substituting (1.2) into (1.1) and differentiating with respect to t we obtain the differential equation governing the wave motion:

$$\frac{\partial \varphi}{\partial t} + \boldsymbol{V} \cdot \nabla \varphi = 0, \tag{1.3}$$

where:

$$\boldsymbol{V} = \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{x}(t), \tag{1.4}$$

is the *ray velocity* of the point of the wave-front and $\cdot$ means the scalar product in the physical space. Assuming $\nabla \varphi \neq 0$, the eq. (1.3) can be written also in the equivalent form:

$$\frac{\partial \varphi}{\partial t} + V \, |\nabla \varphi| = 0, \tag{1.5}$$

where:

$$V = \boldsymbol{V}\cdot\boldsymbol{n}, \qquad \boldsymbol{n} = \frac{\nabla\varphi}{|\nabla\varphi|}, \tag{1.6}$$

is the normal wave speed of the wave-front at the point $\boldsymbol{x}$ and time t. The first step of our approach consists in wondering whether and at which conditions the wave-front equation (1.5) can be interpreted *also* as a Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t} + H = 0, \tag{1.7}$$

governing motion of some family of particles across the physical space, each one being identified by a suitable choice of the initial conditions on the wave-front at initial time. In order to make possible such an interpretation we may think of the Hamilton generating function S governing motion of the particles as proportional to the function φ according to the relation:

$$S = \alpha\varphi, \tag{1.8}$$

where α is a suitable *universal constant,* which will be seen later to be equal to the reduced Planck constant $\hbar$, in order that the physical interpretation of the theory is consistent with quantum mechanics. Immediately one realizes that (1.5) and (1.7) identify if the Hamiltonian of the particle is given by:

$$H = V\,|\nabla S|. \tag{1.9}$$

Since, according to the theory of Hamilton-Jacobi:

$$\boldsymbol{p} = \nabla S\ , \qquad E = H, \tag{1.10}$$

(where $\boldsymbol{p}$ is the canonical momentum of the particle, and the Hamiltonian H is its generalized energy) from (1.9) it results:

$$E = Vp, \qquad p = |\boldsymbol{p}|. \tag{1.11}$$

It is immediate to see that (1.11-a) is compatible with particle mechanics only in the frame of *special of relativity* if and only if:

$$V = c, \tag{1.12}$$

i.e., when the particle has zero rest mass, so that the correct energy-momentum relation:

$$E = cp, \tag{1.13}$$

is obtained. The Hamilton equations (where $\partial/\partial\boldsymbol{x}$ means the gradient operator respect to the co-ordinates and $\partial/\partial\boldsymbol{p}$ is the gradient operator respect to the momentum components):

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -\frac{\partial H}{\partial \boldsymbol{x}}, \qquad \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \frac{\partial H}{\partial \boldsymbol{p}}, \tag{1.14}$$

which here become:

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = c\boldsymbol{n}, \tag{1.15}$$

describe the motion of a massless particle traveling at the speed of light c along a straight line of direction $\boldsymbol{n}$:

$$\boldsymbol{n} = \frac{\boldsymbol{p}}{p}. \tag{1.16}$$

We emphasize that the wave-front equation (1.5), which we have identified with the Hamilton-Jacobi equation (1.7), properly involves motion of an entire family of identical particles, traveling along the wave rays. Each particle starts, at $t = 0$, from some point $\boldsymbol{x}_0$ (initial condition) on the initial wave-front of equation $\varphi(0, \boldsymbol{x}_0)$ and reaches at the time t the position $\boldsymbol{x}(t)$ on the actual wave-front of equation $\varphi(t, \boldsymbol{x}(t)) = 0$.

Even if the identification of wave-front and particle Hamilton-Jacobi equations arose in a very natural way, now two non-trivial problems emerge.

1. The problem of interaction of the particle with external fields, and

2. The problem of the non-vanishing particle rest mass.

The problem of vanishing particle rest mass, well known within the *standard model* of elementary particles appears here naturally as a consequence of our wave-particle unification approach. Starting from chapter 6 we will see how to solve it within a unified field theory.

In the next sections we will examine both questions and we will see how they may be solved if a space with more than 3 dimensions is introduced.

1.2.2 Second Step – Interaction of Particles with Fields

In particle mechanics interactions of particles with fields are usually treated adding a potential V to the Hamiltonian of a free particle:

$$H = H_{\text{free}} + V_{\text{int}}. \tag{1.17}$$

Manifestly a similar way of dealing with interactions appears to immediately destroy any possible identification of the wave-front equation (1.5) with the Hamilton-Jacobi equation (1.7) governing particle motion. Perhaps this may be one of the reasons why such identification was not conceived until now, resulting incompatible with Schrödinger equation which is non-relativistic.

An elegant alternative to the potential V, as suggested by *general relativity,* is provided by the introduction of a suitable metric within the physical space. In fact, if we endow the 3-*dimensional* space with a metric, here defined by the tensor of contravariant components $g_{[3]}^{jk}$, we obtain the result of describing the interaction of a particle by means of a *gravitational field* without altering the form (1.9) of the Hamiltonian H. Simply, now the modulus of particle momentum is evaluated taking into account the metric, avoiding any potential V. In fact at those conditions it results:

$$E = cp, \qquad p = \sqrt{g_{[3]}^{jk} p_j p_k}, \qquad j, k = 1, 2, 3, \tag{1.18}$$

where:

$$p_i = \frac{\partial S}{\partial x^i}, \qquad i = 1, 2, 3, \tag{1.19}$$

and signature of $\left(g_{[3]}^{jk}\right)$ here is $(+,+,+)$ so that the line interval is $\mathrm{d}s^2 = g_{jk}^{[3]} \mathrm{d}x^i dx^k$, where $g_{jk}^{[3]}$ are the covariant components of the metric. Then the Hamilton equations become:

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{c}{2p} g_{[3],i}^{jk}\, p_j p_k, \tag{1.20}$$

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = c\, g_{[3]}^{ik}\, n_k, \tag{1.21}$$

being:

$$n_k = \frac{p_k}{p}. \tag{1.22}$$

It is straightforward to realize that the present approach to wave-particle mechanics strongly suggests that, in order to introduce all the fundamental fields known in physics – *i.e.* the *electromagnetic, weak* and *strong interaction fields,* beside the *gravitational* one – more than three space dimensions need to be involved, in order to avoid potentials which would break the correspondence between wave-front and particle Hamilton-Jacobi equations of motion. We will be concerned with this problem starting from chapter 5. It is remarkable, as we will see in the following subsection, that the same assumption of higher space dimensionality is required also in order that particles may acquire a non-vanishing rest mass.

Incidentally we observe that introduction of higher space dimensionality apparently seems to avoid the need of introducing an additional scalar boson field as previewed by Higgs mechanism in the context of quantum field theory. In effect it is not so, since the vector fields living in the *extra* space dimensions will appear as scalar fields into the physical 3-*dimensional sub*-space and a sort of field confinement will be required in order that the extra co-ordinates are not observable in the ordinary physical space. The scalar boson will appear just as a gauge function for the vector potentials of the interaction and matter fields. This matter will be examined in chapter 6.

1.2.3 Third Step – Non-Vanishing Rest Mass

We have seen in $\S1.2.1$ that the identification of wave-front and Hamilton-Jacobi particle equations of motion requires that the particle itself travels at the speed of light c across the 3-*dimensional* physical space and then its rest mass is necessarily zero.

A simple way to introduce non-zero rest mass of particles, sometimes exploited in literature (see, *e.g.,* [27, 28], [30, 31] and related references) is provided by the introduction of space *extra* dimensions, so that new co-ordinates and new momentum components will be available. Then rest mass may arise from the contribution of the *extra* components of momentum. In fact, if we assume, *e.g.,* that at least one *extra* space dimension exists within a flat space-time, the energy (1.13) can be written in the form:

$$E = c\sqrt{\boldsymbol{p}^2 + p_4^2}, \tag{1.23}$$

where:

$$p_4 = \frac{\partial S}{\partial x^4}. \tag{1.24}$$

Then we would obtain the energy of a particle of rest mass m:

$$E = c\sqrt{\boldsymbol{p}^2 + m^2c^2}, \tag{1.25}$$

if we are able to interpret the *extra* momentum component p_4 as a contribution arising from a rest mass:

$$m = \frac{p_4}{c}. \tag{1.26}$$

The latter condition is equivalent to require that:

$$\frac{\partial S}{\partial x^4} = mc, \tag{1.27}$$

and integrating:

$$S = S^{[3]}(t, x^i) + mcx^4, \tag{1.28}$$

where $S^{[3]}$ is the Hamilton generating function governing motion in the 3-*dimensional* space and the particle mass m is assumed to be a constant.

Adding dimensions to space involves new problems to be examined within the theory, as *e.g.,* that of confinement of the new unobservable co-ordinates and that of the particle rest mass constance. In particular the problem of mass constance results soon to be related to a suitable choice of the arbitrary non-vanishing scale factor for the function φ. In fact if we rescale φ of a non-vanishing dimensionless factor f according to the rule:

$$\overline{\varphi} = f\varphi, \tag{1.29}$$

into (1.28) we have:

$$\alpha f\varphi = S^{[3]}(t, x^i) + mcx^4, \tag{1.30}$$

which determines f as:

$$f = \frac{1}{\alpha\varphi}\big[S(t, x^i) + mcx^4\big], \tag{1.31}$$

for any x^i, t external to the wave front (*i.e.,* when $\varphi \neq 0$).

As we will see later in chapter 6 the previous choice of f will be related to the gauge functions of the vector fields governing matter and interactions.

1.3 Conclusion

In the present chapter we have seen, following a heuristic approach, a possibile and interesting way to unify the concepts of *wave ray* and *particle trajectory* together with the respective time *evolution laws.*

Since the identification of the Hamilton-Jacobi equations governing wave-front propagation and particle motion requires, according to relativity theory that the particles travel at the speed of light c, so resulting massless, we suggested a manner to obtain a non null particle rest mass by introducing additional space dimensions. Interaction of particles with a gravitational field has also be provided thanks to a suitable choice of the metric of space.

In the following two chapters we will present a more general and precise development of the ideas we have just sketched until now.

Chapter 2

Wave-Particles in V^n (Synchronous Frame)

Abstract
In this chapter we develop in a more rigorous way the approach we presented heuristically until now. Wave and particle mechanics, governed by the same *p.d.e.* will be treated within an n*-dimensional* space-time V^n in a synchronous frame. We will see how the well known Einstein-Planck and De Broglie relations arise naturally following our unification method. Moreover we will investigate the problem of looking for a suitable Lagrangian field dynamics leading to wave-particles traveling at the speed of light across the higher dimensional space. Klein-Gordon and Dirac equations will be naturally obtained.

2.1 Introduction

In the first part of the present chapter (§2.2) we will develop, in a more rigorous way some of the heuristic ideas we have just suggested in the previous chapter. In particular we will be concerned with wave-particle mechanics within an n-*dimensional* space-time manifold V^n in a *synchronous* co-ordinate frame.

In the second part of the chapter (§2.3) we will introduce the problem of looking for the properties which are required to some suitable field to describe wave-particles as solutions (see [27, 28]).

A completely covariant theory, independent of the co-ordinate choice, will be presented in the next chapter 3.

2.2 Waves and Particle Dynamics

Let us start considering an n-*dimensional* ($n > 4$) real differentiable manifold V^n, modeling a space-time endowed with a symmetric metric $\boldsymbol{g}$ the signature of which is $(+, -, \cdots, -)$ and a torsionless connection $\boldsymbol{\Gamma}$. On V^n we represent any system of curvilinear co-ordinates with $x^{\bar{\mu}}, \bar{\mu} = 0, \bar{i}$, with $\bar{i} = 1, 2, \cdots, n-1$. The physically observable space-time is then described by a 4-*dimensional* sub-manifold V^4 on which the observable co-ordinates are labelled by the indices $\mu = 0, i$, where $i = 1, 2, 3$, while we will label the remaining co-ordinates $x^4, x^5, \cdots, x^{n-1}$ by underlined Latin indices $\underline{i} = 4, 5, \cdots, n-1$.

For the sake of simplicity we adopt, here, the *synchronous gauge*:

$$g_{00} = 1, \qquad g_{0\bar{i}} = g_{\bar{i}0} = 0, \tag{2.1}$$

i.e., a co-ordinate choice which ensures time synchronization for all the observers, implying:

$$x^0 = ct \qquad \text{(time synchronization)}. \tag{2.2}$$

An explicitly covariant formulation, with no assumption on the co-ordinate gauge, requires a more sophisticated technique and we prefer to expose it later, in a dedicated chapter (see chapter 3).

Let us now consider any differentiable real valued function $\varphi\left(t, x^{\bar{i}}\right)$, which we can always assume to be dimensionless, such that:

$$\tfrac{1}{c^2}\left(\frac{\partial \varphi}{\partial t}\right)^2 + g^{\bar{j}\bar{k}}\,\frac{\partial \varphi}{\partial x^{\bar{j}}}\,\frac{\partial \varphi}{\partial x^{\bar{k}}} \leq 0, \tag{2.3}$$

where, the contravariant components of the metric tensor, in the gauge (2.1), result to be:

$$g^{00} = 1, \qquad g^{0\bar{i}} = g^{\bar{i}0} = 0, \qquad g^{\bar{j}\bar{l}}\,g_{\bar{l}\bar{k}} = \delta^{\bar{j}}_{\bar{k}}. \tag{2.4}$$

Then the equation:

$$\varphi\left(t, x^{\bar{i}}\right) = 0, \tag{2.5}$$

may be interpreted as the (*time-like* or *light-like*) world sheet of a wave-front traveling across the $(n-1)$-*dimensional* space. (Some notes on the elements of non-linear wave propagation theory needed here, are presented in Appendix A).

We emphasize that φ is determined by (2.5) except for an arbitrary non vanishing factor, the choice of which, as we know, will be relevant later to ensure particle mass constance.

The trajectory (*ray*) of each point of the wave-front can be described by its parametric equations:

$$x^{\bar{i}} \equiv x^{\bar{i}}(t). \tag{2.6}$$

Substituting (2.6) into (2.5) and differentiating with respect to t we obtain the differential equation governing the wave motion:

$$\frac{\partial \varphi}{\partial t} + V^{\bar{i}} \frac{\partial \varphi}{\partial x^{\bar{i}}} = 0, \tag{2.7}$$

where:

$$V^{\bar{i}} = \frac{\mathrm{d}}{\mathrm{d}t} x^{\bar{i}}(t), \tag{2.8}$$

is the *ray velocity* of the point of the wave-front.

Assuming $\varphi_{,\bar{i}} \neq 0$ (comma denoting, as usual, partial derivative respect to co-ordinates), equation (2.7) can be written also in the equivalent form:

$$\frac{\partial \varphi}{\partial t} + V|\nabla \varphi| = 0, \qquad |\nabla \varphi| = \sqrt{-g^{\bar{j}\bar{k}} \varphi_{,\bar{j}} \varphi_{,\bar{k}}}. \tag{2.9}$$

Here V is the normal wave speed being:

$$V = V^{\bar{i}} n_{\bar{i}}, \qquad n_{\bar{i}} = \frac{\varphi_{,\bar{i}}}{|\nabla \varphi|}, \qquad g^{\bar{j}\bar{k}} n_{\bar{j}} n_{\bar{k}} = -1. \tag{2.10}$$

The first step of our approach consists in wondering whether and at which conditions the wave-front equation (2.9-a) can be interpreted also as a Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t} + H = 0, \tag{2.11}$$

governing motion of a family of particles, each one identified by its initial position $x_0^{\bar{i}}$ along the wave-front at $t = 0$.

In order to make possible such interpretation we require that the Hamilton generating function of the particle dynamics is directly proportional to the function φ according to the relation:

$$S = \alpha \varphi, \tag{2.12}$$

where α is a dimensional (positive) universal constant which, later will result equal to Planck reduced constant $\hbar$, according to a physical interpretation of the theory (see §2.2.2).

Then we can write (2.9-a) in the form (2.11) if the Hamiltonian of the particle is required to be:

$$H = V\,|\nabla S|, \qquad |\nabla S| = \sqrt{-g^{\bar{j}\bar{k}}\,S_{,\bar{j}}\,S_{,\bar{k}}}. \tag{2.13}$$

Since, according to the theory of Hamilton-Jacobi:

$$p_{\bar{i}} = S_{,\bar{i}}\ , \qquad E = H, \tag{2.14}$$

(where $p_{\bar{i}}$ is the canonical momentum of the particle, and the Hamiltonian H is the generalized energy), from (2.13) it results:

$$E = V\sqrt{-g^{\bar{j}\bar{k}}\,p_{\bar{j}}\,p_{\bar{k}}}. \tag{2.15}$$

It is immediate to see that (2.15) is compatible with relativity theory if and only if:

$$V = c, \tag{2.16}$$

so that the particle is necessarily required to have zero rest mass. In fact at those conditions it results:

$$E = cp, \qquad p = \sqrt{-g^{\bar{j}\bar{k}}\,p_{\bar{j}}\,p_{\bar{k}}}. \tag{2.17}$$

2.2.1 The Hamilton Equations

The Hamilton equations:

$$\frac{\mathrm{d}p_{\bar{i}}}{\mathrm{d}t} = -\frac{\partial H}{\partial x^{\bar{i}}}, \qquad \frac{\mathrm{d}x^{\bar{i}}}{\mathrm{d}t} = \frac{\partial H}{\partial p_{\bar{i}}}, \tag{2.18}$$

result now:

$$\frac{\mathrm{d}p_{\bar{i}}}{\mathrm{d}t} = \frac{c}{2p} g^{\bar{j}\bar{k}}_{\;\;,\bar{i}}\, p_{\bar{j}}\, p_{\bar{k}}\,, \tag{2.19}$$

$$\frac{\mathrm{d}x^{\bar{i}}}{\mathrm{d}t} = -cn^{\bar{i}}, \tag{2.20}$$

where:

$$n^{\bar{i}} = \frac{p^{\bar{i}}}{p}, \quad p^{\bar{j}} = g^{\bar{j}\bar{k}}\, p_{\bar{k}}. \tag{2.21}$$

One realizes that (2.19) is also equivalent to the geodesic condition:

$$\frac{\mathrm{d}p_{\bar{i}}}{\mathrm{d}t} - \Gamma^{\bar{k}}_{\bar{i}\bar{j}}\, p_{\bar{k}}\, \frac{\mathrm{d}x^{\bar{j}}}{\mathrm{d}t} = 0, \tag{2.22}$$

which, thanks to (2.20), becomes:

$$\frac{\mathrm{d}p_{\bar{i}}}{\mathrm{d}t} + \frac{c}{p}\Gamma^{\bar{k}}_{\bar{i}\bar{j}}\, p^{\bar{j}}\, p_{\bar{k}} = 0. \tag{2.23}$$

In fact:

$$\tfrac{1}{2} g^{\bar{j}\bar{k}}_{\;\;,\bar{i}}\, p_{\bar{j}}\, p_{\bar{k}} \equiv -\Gamma^{\bar{k}}_{\bar{i}\bar{j}}\, p_{\bar{k}}\, p^{\bar{j}},$$

being:

$$g^{\bar{j}\bar{k}}_{\;\;;\bar{i}} \equiv g^{\bar{j}\bar{k}}_{\;\;,\bar{i}} + \Gamma^{\bar{j}}_{\bar{i}\bar{m}}\, g^{\bar{m}\bar{k}} + \Gamma^{\bar{k}}_{\bar{i}\bar{m}}\, g^{\bar{j}\bar{m}} = 0, \tag{2.24}$$

because of the metricity of $g^{\bar{j}\bar{k}}$ (semicolon denoting covariant derivative), where:

$$\Gamma^{\bar{k}}_{\bar{i}\bar{j}} = \tfrac{1}{2} g^{\bar{k}\bar{m}}\,(g_{\bar{j}\bar{m},\bar{i}} + g_{\bar{i}\bar{m},\bar{j}} - g_{\bar{i}\bar{j},\bar{m}}). \tag{2.25}$$

Therefore we may conclude that the differential equation governing wave motion may be identified with the Hamilton-Jacobi equation governing motion of a massless particle traveling at the speed of light across the $(n-1)$-*dimensional* space, provided that the wave itself travels at the speed of light.

Now we can split the $(n-1)$-*vectors* $(p_{\bar{i}}), (x^{\bar{i}})$ into their components onto the physical space p_i, x^i and the *extra* components $p_{\underline{i}}, x^{\underline{i}}$ (remember that, according to our convention $i = 1, 2, 3$ and $\underline{i} = 4, 5, \cdots, n-1$).

We have:

$$\begin{aligned} \frac{\mathrm{d}p_i}{\mathrm{d}t} &= \frac{c}{2p} g^{jk}_{\ \ ,i}\, p_j p_k + \frac{c}{2p} g^{\underline{j}\underline{k}}_{\ \ ,i}\, p_{\underline{j}} p_{\underline{k}} + \frac{c}{p} g^{j\underline{k}}_{\ \ ,i}\, p_j p_{\underline{k}}, \\ \frac{\mathrm{d}x^i}{\mathrm{d}t} &= -c\frac{p^i}{p}, \end{aligned} \tag{2.26}$$

$$\begin{aligned} \frac{\mathrm{d}p_{\underline{i}}}{\mathrm{d}t} &= \frac{c}{2p} g^{jk}_{\ \ ,\underline{i}}\, p_j p_k + \frac{c}{2p} g^{\underline{j}\underline{k}}_{\ \ ,\underline{i}}\, p_{\underline{j}} p_{\underline{k}} + \frac{c}{p} g^{j\underline{k}}_{\ \ ,\underline{i}}\, p_j p_{\underline{k}}, \\ \frac{\mathrm{d}x^{\underline{i}}}{\mathrm{d}t} &= -c\frac{p^{\underline{i}}}{p}. \end{aligned} \tag{2.27}$$

We can interpret:

$$m = \tfrac{1}{c}\sqrt{-(2g^{\underline{j}k}p_k + g^{\underline{j}\underline{k}}p_{\underline{k}})p_{\underline{j}}}, \tag{2.28}$$

as a constant rest mass of some particle provided that we assume:

$$-(2g^{\underline{j}k}p_k + g^{\underline{j}\underline{k}}p_{\underline{k}})p_{\underline{j}} = \textit{non-negative constant}. \tag{2.29}$$

Such a choice is always possible, mathematically, since $p_{\bar{i}} = \alpha\varphi_{,\bar{i}}$ and φ is defined except for a non vanishing factor according to (2.5). It is interesting to observe that this factor is not influent on wave propagation since the wave-front remains unmodified whatever it is, but it is relevant on particle propagation in order to ensure rest mass constance. It appears as a sort of gauge condition on the function φ. More investigations in order to understand a physical "mechanism" ensuring such a constance of mass will be exploited in chapter 6, in Part II.

Then we gain the Hamiltonian of a massive particle:

$$H = c\sqrt{m^2c^2 + \vec{p}^{\,2}}, \qquad \vec{p}^{\,2} = -g^{ik}p_i p_k. \tag{2.30}$$

We observe that when g_{jk} depends only on the observable variables t, x^i, relation (2.26-a) identifies with the equation of the geodesic trajectory of a particle of rest mass m crossing the 3-*dimensional* physical space in presence of a gravitational field, while $g_{\underline{j}\,\underline{k}}, g_{j\underline{k}}$, could represent possible non-gravitational fields in the frame of a filed theory in more than four space-time dimensions like a Kaluza-Klein type theory or some alternative to this latter.

2.2.2 De Broglie and Einstein-Planck Relations

The Hamilton generating function S, proportional to the function φ characterizing the wave-front, can now be evaluated by integration of (2.7), in which the ray velocity $V^{\bar{i}}$ has the direction of the normal vector of components $n^{\bar{i}}$, resulting:

$$V^{\bar{i}} = -cn^{\bar{i}}. \tag{2.31}$$

Then the equation to be integrated becomes:

$$\frac{\partial\varphi}{\partial t} - cn^{\bar{i}}\frac{\partial\varphi}{\partial x^{\bar{i}}} = 0. \tag{2.32}$$

The solution is any function of type:

$$f \equiv f\left(n_{\bar{i}}x^{\bar{i}} - ct\right). \tag{2.33}$$

In fact it results:

$$\frac{\partial\varphi}{\partial t} = -f'c, \qquad \frac{\partial\varphi}{\partial x^{\bar{i}}} = f'n_{\bar{i}}, \qquad n_{\bar{i}} = \frac{p_{\bar{i}}}{p},$$

$p_{\bar{i}}, x^{\bar{i}}$ being independent canonical variables and *prime* denoting here the derivative respect to the whole function argument.

In detail we have:

$$\varphi_{,t} - cn^{\bar{i}}\varphi_{,\bar{i}} = f' n_{\bar{j};t} x^{\bar{j}} - f'c - f' cn^{\bar{i}} n_{\bar{j};\bar{i}} x^{\bar{j}} - f' cn^{\bar{i}} n_{\bar{i}} \equiv f' n_{\bar{j};t} x^{\bar{j}} - f' cn^{\bar{i}} n_{\bar{j};\bar{i}} x^{\bar{j}},$$

being $n^{\bar{i}} n_{\bar{i}} = -1$. Since the wave propagates along a ray of equation $x^{\bar{j}} = cn^{\bar{j}} t$ it follows:

$$n_{\bar{j};t} x^{\bar{j}} - cn^{\bar{i}} n_{\bar{j};\bar{i}} x^{\bar{j}} \equiv n_{\bar{j};t} n^{\bar{j}} ct - cn^{\bar{i}} n_{\bar{j};\bar{i}} n^{\bar{j}} ct = 0,$$

resulting $n_{\bar{j};t} n^{\bar{j}} = 0, n_{\bar{j};\bar{i}} n^{\bar{j}} = 0$.

It follows:

$$\frac{\partial \varphi}{\partial t} - cn^{\bar{i}} \frac{\partial \varphi}{\partial x^{\bar{i}}} \equiv -(1 + n^{\bar{i}} n_{\bar{i}}) f' c = 0,$$

the norm of $(n^{\bar{i}})$ being the negative unit ($n^{\bar{i}} n_{\bar{i}} = -1$).

Such kind of solution is what, in non-linear wave propagation theory, generalizes a *plane wave* in linear wave theory and it is known in literature as a *simple wave* (see, *e.g.,* [8]).

Being $S = \alpha\varphi$, it results also:

$$S = \alpha f(n_{\bar{i}} x^{\bar{i}} - ct). \tag{2.34}$$

Then, thanks to (2.14), the relation between the canonical momentum of the particle and the wave-front solution, during wave-particle motion, becomes:

$$p_{\bar{i}} = \alpha f' n_{\bar{i}}, \tag{2.35}$$

and the relation between the Hamiltonian of the particle and the wave-front solution results:

$$H \equiv E = \alpha c f'. \tag{2.36}$$

If we introduce the following quantities related to wave propagation:

$$k_{\bar{i}} = f' n_{\bar{i}}, \qquad \omega = c f', \tag{2.37}$$

and we choose α equal to the reduced Planck constant $\hbar$, the previous equations arise as a generalization of well known quantum mechanical relations.

In fact we obtain:

$$\begin{aligned} p_{\bar{i}} &= \hbar k_{\bar{i}} \qquad \text{(De Broglie relation)}, \\ E &= \hbar \omega \qquad \text{(Einstein-Planck relation)}. \end{aligned} \tag{2.38}$$

In the special case of a periodic wave it is immediate to recognize the quantities $k_{\bar{i}}, \omega$, as the usual *wave number* vector and *frequency,* of which the definitions (2.37) represent a generalization to non-linear waves.

In general it is convenient to choose $k_{\underline{i}} x^{\underline{i}} - \omega t$ as argument of wave solutions, since it results simply:

$$\varphi = k_{\underline{i}} x^{\underline{i}} - \omega t, \tag{2.39}$$

because of compatibility with (2.38), resulting now $\varphi' \equiv f' = 1$.

2.3 Wave Dynamics and Field Equations

The preliminary level of the problem of wave dynamics consists in determining a class of *fields* governed by *field equations* which yield the wave solutions equivalent to the solutions to the equation of Hamilton-Jacobi for particles, as examined in the previous sections. Of course the main condition required to the system of field equations is that it leads to wave solutions traveling with the speed of light across an

$(n-1)$-*dimensional* space. But that condition alone is too loose to characterize a physically meaningful class of field equations. So we need some more reasonable assumptions. Therefore we will require the following ones:

1. The system of field equations is required to be *Lagrangian.* This condition is usual for physical fields and needs no special explanation.
2. The production term of the system must be zero (at least in correspondence to the simple wave solutions when the system is non-linear). This second assumption is required in order to provide *regular solutions* to the system as *simple waves* are (see Appendix A). Moreover it ensures that the particles travel along the rays with characteristic speeds. In principle also *discontinuity waves* could be considered, but if we want to be able to compare the results with quantum mechanical ones we need regular solutions which can be expanded into Fourier series.
3. The normal speed of all simple waves needs to be equal to the speed of light c. This is just the condition we have previously determined.

2.3.1 The Lagrangian System

Let us consider a candidate field $\boldsymbol{\phi}$, which in general may be a complex column vector belonging to an N-*dimensional* Euclidean complex space, and is assumed to be a set of regular functions of $t, x^{\bar{i}}$, invariant respect to any regular co-ordinate transformation. And let:

$$\mathcal{L} \equiv \sqrt{|g|}\;\ell\big(\boldsymbol{v}, \boldsymbol{v}^{+}, \boldsymbol{w}_{\bar{i}}, \boldsymbol{w}_{\bar{i}}^{+}\big), \qquad \boldsymbol{v} = \frac{\partial \boldsymbol{\phi}}{\partial t}, \quad \boldsymbol{w}_{\bar{i}} = \frac{\partial \boldsymbol{\phi}}{\partial x^{\bar{i}}}, \tag{2.40}$$

be a Lagrangian density governing the field dynamics (where $^{+}$ denotes transposed complex conjugation). Such a Lagrangian is supposed to depend only on $\boldsymbol{v}, \boldsymbol{w}_{\bar{i}}$, in order to fulfill the assumption (2), or, at least to involve $\boldsymbol{\phi}$ in such a way that the production term $\partial\mathcal{L}/\partial\boldsymbol{\phi}$ vanishes in correspondence to the simple wave solutions, we are interested in, or at least it results to be negligible under physically suitable conditions. (Here and in the following, notations like $\partial/\partial\boldsymbol{\phi}$ denote the gradient operator respect to the components of the vector $\boldsymbol{\phi}$). So, in the following of the present section we will drop it. Then the system of first order field equations results:

$$\frac{\partial}{\partial t}\Big(\sqrt{|g|}\,\frac{\partial\ell}{\partial\boldsymbol{v}^+}\Big) + \frac{\partial}{\partial x^{\bar{i}}}\Big(\sqrt{|g|}\,\frac{\partial\ell}{\partial\boldsymbol{w}^+_{\bar{i}}}\Big) = 0, \tag{2.41}$$

$$\frac{\partial\boldsymbol{w}_{\bar{i}}}{\partial t} - \frac{\partial\boldsymbol{v}}{\partial x^{\bar{i}}} = 0, \tag{2.42}$$

together with its complex conjugate. We are looking for simple wave solutions, *i.e.,* functions $\boldsymbol{\phi}\big(\varphi(t, x^{\underline{i}})\big)$ which depend on $t, x^{\underline{i}}$ by means of an argument function φ.

On applying the correspondence rules:

$$\begin{aligned} &\frac{\partial}{\partial t}(\cdot) \longrightarrow -\lambda\frac{\partial}{\partial\varphi}(\cdot) \equiv -\lambda(\cdot)', \\ &\frac{\partial}{\partial x^{\bar{i}}}(\cdot) \longrightarrow n_{\bar{i}}\frac{\partial}{\partial\varphi}(\cdot) \equiv n_{\bar{i}}(\cdot)', \end{aligned} \tag{2.43}$$

prime denoting here differentiation respect to the argument φ, we obtain, according to wave propagation theory, the algebraic system for the simple waves eigenvalue problem:

$$-\lambda\Big(\sqrt{|g|}\,\frac{\partial\ell}{\partial\boldsymbol{v}^+}\Big)' + n_{\bar{i}}\Big(\sqrt{|g|}\,\frac{\partial\ell}{\partial\boldsymbol{w}^+_{\bar{i}}}\Big)' = 0, \tag{2.44}$$

$$-\lambda\boldsymbol{w}'_{\bar{i}} - n_{\bar{i}}\boldsymbol{v}' = 0, \tag{2.45}$$

in which λ is the *characteristic normal speed* of the simple waves and the unknown field variables $\boldsymbol{v}, \boldsymbol{w}_{\bar{i}}$ are functions of φ. We point out that even if the wave propagation theory we are applying here was originally carried out in the context of real valued fields, its extension to complex fields is straightforward (see *e.g.,* [26]).

Simple manipulations on (2.44)-(2.45), after eliminating $\boldsymbol{w}_{\bar{i}}$ by direct substitution, lead to:

$$\left\{\lambda^2 \frac{\partial^2 \ell}{\partial \boldsymbol{v}^+ \partial \boldsymbol{v}} g^{\bar{i}\bar{j}} - \lambda\left(n^{\bar{i}} \frac{\partial^2 \ell}{\partial \boldsymbol{v}^+ \partial \boldsymbol{w}_{\bar{j}}} + \frac{\partial^2 \ell}{\partial \boldsymbol{w}_{\bar{i}}^+ \partial \boldsymbol{v}} n^{\bar{j}}\right) - \right.$$
$$\left. - \frac{\partial^2 \ell}{\partial \boldsymbol{w}_{\bar{i}}^+ \partial \boldsymbol{w}_{\bar{j}}}\right\} n_{\bar{i}} n_{\bar{j}} \boldsymbol{v}' = 0. \tag{2.46}$$

Since the Lagrangian density cannot depend on the wave-front geometry, *i.e.,* on $n_{\bar{i}}$, the following conditions must hold:

$$\lambda^2 \frac{\partial^2 \ell}{\partial \boldsymbol{v}^+ \partial \boldsymbol{v}} g^{\bar{i}\bar{j}} - \frac{\partial^2 \ell}{\partial \boldsymbol{w}_{\bar{i}}^+ \partial \boldsymbol{w}_{\bar{j}}} = 0, \tag{2.47}$$

$$\frac{\partial^2 \ell}{\partial \boldsymbol{v}^+ \partial \boldsymbol{w}_{\bar{j}}} = 0 \quad \Longleftrightarrow \quad \frac{\partial^2 \ell}{\partial \boldsymbol{w}_{\bar{i}}^+ \partial \boldsymbol{v}} = 0. \tag{2.48}$$

The latter condition implies that the fields $\boldsymbol{v}, \boldsymbol{w}_{\bar{i}}$, and their respective conjugates, are decoupled, resulting:

$$\ell = \ell_1(\boldsymbol{v}, \boldsymbol{v}^+) + \ell_2(\boldsymbol{w}_{\bar{i}}, \boldsymbol{w}_{\bar{i}}^+). \tag{2.49}$$

Substitution into the former equation yields:

$$\lambda^2 \frac{\partial^2 \ell_1}{\partial \boldsymbol{v}^+ \partial \boldsymbol{v}} g^{\bar{i}\bar{j}} - \frac{\partial^2 \ell_2}{\partial \boldsymbol{w}_{\bar{i}}^+ \partial \boldsymbol{w}_{\bar{j}}} = 0. \tag{2.50}$$

Thanks to our assumption (3), the normal speeds of simple wave propagation are to be set equal the speed of light, and therefore (2.47) becomes:

$$c^2 \frac{\partial^2 \ell_1}{\partial \boldsymbol{v}^+ \partial \boldsymbol{v}} g^{\bar{i}\bar{j}} = \frac{\partial^2 \ell_2}{\partial \boldsymbol{w}_{\bar{i}}^+ \partial \boldsymbol{w}_{\bar{j}}}. \tag{2.51}$$

The Hessian matrices of ℓ_1, ℓ_2 are manifestly independent of the fields $\boldsymbol{v}, \boldsymbol{w}_{\bar{i}}$, so we may write them as:

$$\frac{\partial^2 \ell_1}{\partial \boldsymbol{v}^+ \partial \boldsymbol{v}} = \tfrac{1}{c^2}\, \mathsf{a}, \tag{2.52}$$

$$\frac{\partial^2 \ell_2}{\partial \boldsymbol{w}^+_{\bar{i}} \partial \boldsymbol{w}_{\bar{j}}} = g^{\bar{i}\bar{j}}\, \mathsf{a}, \tag{2.53}$$

where a is a non singular Hermitean matrix, independent of the fields. From those conditions the form of the Lagrangian density is determined to be:

$$\mathcal{L} = \tfrac{1}{2}\sqrt{|g|}\Big(g^{\bar{i}\bar{j}}\, \boldsymbol{w}^+_{\bar{i}}\, \mathsf{a}\boldsymbol{w}_{\bar{j}} + \tfrac{1}{c^2}\boldsymbol{v}^+ \mathsf{a}\boldsymbol{v}\Big). \tag{2.54}$$

The Euler-Lagrange equations are given by:

$$\frac{\partial}{\partial t}\Big(\sqrt{|g|}\; \mathsf{a}\boldsymbol{v}\Big) + \frac{\partial}{\partial x^{\bar{i}}}\Big(c^2 \sqrt{|g|}\; g^{\bar{i}\bar{j}}\, \mathsf{a}\boldsymbol{w}_{\bar{j}}\Big) = 0, \tag{2.55}$$

or:

$$\big(\mathsf{a}\boldsymbol{v}\big)_{;0} + \big(c\, g^{\bar{i}\bar{j}}\, \mathsf{a}\boldsymbol{w}_{\bar{j}}\big)_{;\bar{i}} = 0, \tag{2.56}$$

where $_{;0}$ denotes the covariant derivative respect to $x^0 = ct$. We observe that a, beside being independent of the fields $\boldsymbol{v}, \boldsymbol{w}_{\bar{i}}$, must be independent also of $t, x^{\bar{i}}$, otherwise a non vanishing production term would arise into the field equations. Then, since a is a non singular matrix, taking account of the metricity condition ($g^{\bar{i}\bar{j}}_{\;;\bar{k}} = 0$) and symmetries, the field equations result simply:

$$\boldsymbol{v}_{;0} + c\, g^{\bar{j}\bar{i}}\, \boldsymbol{w}_{\bar{j}\,;\bar{i}} = 0, \quad c\, \boldsymbol{w}_{\bar{i}\,;0} - \boldsymbol{v}_{;\bar{i}} = 0. \tag{2.57}$$

The coefficient matrix a disappears in the equations and its role becomes irrelevant in the Lagrangian. So it is not a restriction to choose:

$$\mathsf{a} = \varkappa\, \mathsf{I}, \tag{2.58}$$

with $\varkappa$ a suitable dimensional constant and $\boldsymbol{I}$ the identity matrix in the field space.

Then the scalar Lagrangian density becomes finally:

$$\ell = \tfrac{1}{2}\varkappa\Big(g^{\bar{i}\bar{j}}\,\boldsymbol{w}_{\bar{i}}^{+}\boldsymbol{w}_{\bar{j}} + \tfrac{1}{c^2}\boldsymbol{v}^{+}\boldsymbol{v}\Big). \tag{2.59}$$

We observe that the algebraic system which determines the simple waves, arising from (2.57), which is given by:

$$\tfrac{1}{c}\lambda\boldsymbol{v}' - cn^{\bar{i}}\,\boldsymbol{w}_{\bar{i}}' = 0, \quad \lambda\boldsymbol{w}_{\bar{i}}' + n_{\bar{i}}\,\boldsymbol{v}' = 0, \tag{2.60}$$

yields, through direct substitution:

$$(\lambda^2 - c^2)\,\boldsymbol{v}' = 0, \tag{2.61}$$

from which the characteristic speeds become:

$$\lambda = \pm c, \tag{2.62}$$

as expected. Such values imply, significantly, that the energy of the particle associated to wave propagation, thanks to (2.15), taking into account that $V = \lambda$, results to be:

$$E = \pm\sqrt{m^2c^4 + c^2p^{\,2}}. \tag{2.63}$$

It is remarkable that, corresponding to any simple wave-front traveling at normal speed $+c$, there exists an identical wave-front traveling in the opposite sense, *i.e.,* at speed $-c$, as it is usual in wave propagation. So, corresponding to a particle of energy $+|E|$, associated with the former wave, there exists another particle of negative energy $-|E|$ (*anti*-particle), associated to the latter wave. Such circumstance, as it is well known, was noticed for the first time by P.A.M. Dirac (see Dirac original papers [14, 15]), examining the solutions of his famous equation. According to our present approach the result arises naturally as a consequence of wave particle correlation expressed by (2.15), being $V=\lambda$, and the assumption that the system of field equations is Lagrangian.

2.3.2 Hamiltonian Density

Starting from the Lagrangian density given by (2.54), (2.59) we can evaluate the Hamiltonian density:

$$\mathcal{H} = \boldsymbol{v}^{+}\frac{\partial \mathcal{L}}{\partial \boldsymbol{v}} - \mathcal{L}, \tag{2.64}$$

of the field $\boldsymbol{\phi}$ in terms of its derivatives $\boldsymbol{v}, \boldsymbol{w}_{\bar{i}}$.

We obtain soon:

$$\mathcal{H} = \tfrac{1}{2}\varkappa\sqrt{|g|}\Big(\tfrac{1}{c^2}\boldsymbol{v}^{+}\boldsymbol{v} - g^{\bar{i}\bar{j}}\,\boldsymbol{w}_{\bar{i}}^{+}\boldsymbol{w}_{\bar{j}}\Big). \tag{2.65}$$

In particular, in correspondence to a periodic wave-particle solution (*boson*) to the field equations (2.57), like:

$$\begin{aligned} \boldsymbol{v} &= -i\omega\,\boldsymbol{\phi}_0 e^{i(k_{\bar{i}}x^{\bar{i}}-\omega t)} + i\omega\,\boldsymbol{\phi}_0^{+}\,e^{i(k_{\bar{i}}x^{\bar{i}}+\omega t)}, \\ \boldsymbol{w}_{\bar{j}} &= i k_{\bar{j}}\,\boldsymbol{\phi}_0 e^{i(k_{\bar{i}}x^{\bar{i}}-\omega t)} + i k_{\bar{j}}\,\boldsymbol{\phi}_0^{+}\,e^{i(k_{\bar{i}}x^{\bar{i}}+\omega t)}, \end{aligned} \tag{2.66}$$

it results:

$$\mathcal{H} = \tfrac{1}{2}\varkappa\sqrt{|g|}\Big(\tfrac{\omega^2}{c^2} - g^{\bar{i}\bar{j}}\,k_{\bar{i}}\,k_{\bar{j}}\Big)\big(\boldsymbol{\phi}_0^{+}\boldsymbol{\phi}_0 + \boldsymbol{\phi}_0\boldsymbol{\phi}_0^{+}\big). \tag{2.67}$$

Now $\mathcal{H}$ represents the energy density of the field $\boldsymbol{\phi}$ (or equivalently of the fields $\boldsymbol{v}, \boldsymbol{w}_{\bar{i}}$). Since in V^n it results also:

$$\tfrac{E^2}{c^2} + g^{\bar{i}\bar{j}}\,p_{\bar{i}}\,p_{\bar{j}} \equiv \hbar^2\big(\tfrac{\omega^2}{c^2} + g^{\bar{i}\bar{j}}\,k_{\bar{i}}\,k_{\bar{j}}\big) = 0, \tag{2.68}$$

it follows:

$$\mathcal{H} = \tfrac{\varkappa}{c^2}\sqrt{|g|}\omega^2\big(\boldsymbol{\phi}_0^{+}\boldsymbol{\phi}_0 + \boldsymbol{\phi}_0\boldsymbol{\phi}_0^{+}\big). \tag{2.69}$$

Quantization is obtained now introducing the annihilation/creation operators $\mathsf{a}, \mathsf{a}^{+}$ such that:

$$\frac{\sqrt{\varkappa}}{c}\,\omega\boldsymbol{\phi}_0 = \sqrt{\frac{\hbar\omega}{2}}\,\mathsf{a}, \tag{2.70}$$

for which:

$$\mathsf{a}\mathsf{a}^{+} - \mathsf{a}^{+}\mathsf{a} = \mathsf{I}, \tag{2.71}$$

so that we have:

$$\mathcal{H} = \sqrt{|g|}\,\hbar\omega\big(\mathsf{a}^{+}\mathsf{a} + \tfrac{1}{2}\big). \tag{2.72}$$

In presence of many wave-particles $\mathcal{H}$ results to be the summation of each contribution, *i.e.*:

$$\mathcal{H} = \sqrt{|g|}\,\sum_n \hbar\omega_n\big(\mathsf{a}_n^{+}\mathsf{a}_n + \tfrac{1}{2}\big). \tag{2.73}$$

2.3.3 The Klein-Gordon Equation

Let us now study in more detail the system of field equations (2.57).

Second Order Formulation

First of all we observe that we can always replace the fields $\boldsymbol{v}, \boldsymbol{w}_{\bar{\imath}}$, with their original definitions in terms of derivatives of the field $\boldsymbol{\phi}$, according to (2.40). We obtain a system of N second order equations:

$$\boldsymbol{\phi}_{;0;0} + g^{\bar{\imath}\bar{\jmath}}\,\boldsymbol{\phi}_{;\bar{\imath};\bar{\jmath}} = 0, \tag{2.74}$$

instead of the original system of $2N$ first order equations (2.57).

This generalized D'Alembert equation, governing the propagation of the field $\boldsymbol{\phi}$ across the $(n-1)$-*dimensional* space, is equivalent to a

Klein-Gordon equation for the propagation of the same field across the physical 3-*dimensional* space. In fact the solutions for the field, being simple waves, depend on the argument $n_{\bar{i}}x^{\bar{i}} - ct$ or equivalently $k_{\bar{i}}x^{\bar{i}} - \omega t$, being composite functions through φ of the same argument. Now, thanks to (2.38) we have:

$$p_{\bar{i}}x^{\bar{i}} - Et \equiv \hbar(k_{\bar{i}}x^{\bar{i}} - \omega t), \tag{2.75}$$

so that we may express $k_{\bar{i}}x^{\bar{i}} - \omega t$ in terms of $p_{\bar{i}}x^{\bar{i}} - Et$. Therefore we can consider the field $\boldsymbol{\phi}$ as a function of the argument $p_{\bar{i}}x^{\bar{i}} - Et$, rather than $n_{\bar{i}}x^{\bar{i}} - ct$:

$$\boldsymbol{\phi} \equiv \boldsymbol{\phi}(p_{\bar{i}}x^{\bar{i}} - Et). \tag{2.76}$$

Conveniently we split the scalar product in a contribution arising from the ordinary space components $(p_i x^i)$ and the extra-space components $(p_{\underline{i}}x^{\underline{i}})$, obtaining:

$$\boldsymbol{\phi} \equiv \boldsymbol{\phi}(p_i x^i + p_{\underline{i}}x^{\underline{i}} - Et). \tag{2.77}$$

Without affecting time we may choose the space co-ordinates $x^{\underline{i}}$ in such a way that:

$$x^n = x^{\underline{i}}N_{\underline{i}}, \qquad g^{nn} \equiv g^{\underline{j}\underline{k}}N_{\underline{j}}N_{\underline{k}} = -1, \qquad g^{\underline{i}j} = g^{j\underline{i}} = 0, \tag{2.78}$$

$N_{\underline{i}}$ being the unit vector along the direction of the of momentum extra component:

$$N_{\underline{i}} = \frac{p_{\underline{i}}}{\sqrt{-g^{\underline{j}\underline{k}}p_{\underline{j}}p_{\underline{k}}}}, \tag{2.79}$$

which results to be equivalent to:

$$p_{\underline{i}} = mcN_{\underline{i}}, \tag{2.80}$$

thanks to (2.28).

Then the simple wave will travel across a 5-*dimensional* sub-space-time, resulting:

$$\boldsymbol{\phi} \equiv \boldsymbol{\phi}(p_i x^i + mcx^n - Et). \tag{2.81}$$

If we require the additional assumption that $\boldsymbol{\phi}$ belongs to the Hilbert space L^2, as usual in quantum mechanics, so that it can be expanded into Fourier series, we may write:

$$\boldsymbol{\phi} = \sum_{r=-\infty}^{+\infty} \boldsymbol{c}_r e^{\frac{i}{\hbar}\left(p_{ri} x^i + mcx^n - E_r t\right)} \equiv e^{\frac{i}{\hbar}mcx^n} \sum_{r=-\infty}^{+\infty} \boldsymbol{c}_r e^{\frac{i}{\hbar}\left(p_{ri} x^i - E_r t\right)}, \tag{2.82}$$

where the energies result to be:

$$E_r = c\sqrt{|\vec{p}_r|^{\,2} + m^2c^2}, \qquad |\vec{p}_r|^{\,2} = -g^{jk} p_{rj} p_{rk}. \tag{2.83}$$

Now we are able to evaluate the Laplacian:

$$g^{\underline{i}\underline{j}}\, \phi_{;\underline{i};\underline{j}} \equiv g^{nn} \phi_{;n;n} = \tfrac{m^2c^2}{\hbar^2}\phi.$$

Then the field equation (2.74) leads to the Klein-Gordon equation in generalized co-ordinates:

$$\phi_{;0;0} + g^{ij}\phi_{;i;j} + \tfrac{m^2c^2}{\hbar^2}\phi = 0. \tag{2.84}$$

First Order Formulation

A further relevant consideration arises on evaluating the divergence of $\boldsymbol{w}_{\bar{i}}$ in terms of the field variables $\boldsymbol{w}_i$ and $\boldsymbol{w}_n$. Taking into account the previous results we have:

$$\boldsymbol{w}_{\underline{j}} = \boldsymbol{w}_n \delta_{\underline{j}n}, \qquad \boldsymbol{w}_{\underline{j};n} = \tfrac{imc}{\hbar}\boldsymbol{w}_n \delta_{\underline{j}n}. \tag{2.85}$$

It follows into the system (2.57):

$$\boldsymbol{v}_{;0} + c\, g^{ij}\, \boldsymbol{w}_{j;i} + i\tfrac{mc^2}{\hbar}\boldsymbol{w}_n = 0. \tag{2.86}$$

$$c\, \boldsymbol{w}_{i;0} - \boldsymbol{v}_{;i} = 0, \tag{2.87}$$

$$c\, \boldsymbol{w}_{n;0} - \boldsymbol{v}_{;n} = 0. \tag{2.88}$$

This result is equivalent to say that we need al least *one* extra space dimension to introduce the rest mass of a single particle. More dimensions will be required to deal with several kinds of particles as it is the case of elementary particles *standard model.* Then the space-time needs to have five dimensions if the particle travels across a gravitational field or six if the electromagnetic field is added or more if other fundamental fields are present.

2.3.4 The Dirac Equation

In the N-*dimensional* linear space of the field variables $\boldsymbol{w}_{\bar{i}}, \boldsymbol{v}$ it is always possible to find four non singular matrices relating a component of the vectors $\boldsymbol{w}_i, \boldsymbol{v}$ with the vector $\boldsymbol{w}_n$; so we can introduce a vector of matrices $\boldsymbol{\alpha}_i, \boldsymbol{\alpha}$, such that:

$$\boldsymbol{w}_i = -K\, \boldsymbol{\alpha}_i \boldsymbol{w}_n, \qquad \boldsymbol{v} = -K\, c \boldsymbol{\alpha} \boldsymbol{w}_n, \tag{2.89}$$

where K is a suitable constant to be determined. The previous relation implies also:

$$\boldsymbol{\alpha v} + c\boldsymbol{\alpha}^i \boldsymbol{w}_i = -Kc(\boldsymbol{\alpha}^2 + \boldsymbol{\alpha}^i \boldsymbol{\alpha}_i)\boldsymbol{w}_n, \qquad \boldsymbol{\alpha}^i = g^{ij}\, \boldsymbol{\alpha}_i. \tag{2.90}$$

Assuming that $\boldsymbol{\alpha}^2 + \boldsymbol{\alpha}^i \boldsymbol{\alpha}_i$ is non singular it is possible to introduce another set of matrices a^i, a such that:

$$\boldsymbol{w}_n = -\tfrac{1}{Kc}\big(\mathsf{a}\boldsymbol{v} + c\,\mathsf{a}^i \boldsymbol{w}_i\big), \tag{2.91}$$

Combining (2.91) with (2.89) we get:

$$\boldsymbol{w}_n = \big(\mathsf{a}\,\boldsymbol{\alpha} + \mathsf{a}^i \boldsymbol{\alpha}_i\big)\, \boldsymbol{w}_n.$$

And therefore:

$$\boldsymbol{a}\boldsymbol{\alpha} + \boldsymbol{a}^i\boldsymbol{\alpha}_i = \boldsymbol{I}. \tag{2.92}$$

Thanks to (2.91) and taking account of Schwarz condition (holding for the scalar components of the field $\boldsymbol{\phi}$), the field equations (2.57) may be written only in terms of $\boldsymbol{v}, \boldsymbol{w}_i, \boldsymbol{w}_n$. We have:

$$\boldsymbol{v}_{;0} + Kc\, g^{ij}\boldsymbol{w}_{j;i} + Kcg^{nn}\boldsymbol{w}_{n;n} = 0, \tag{2.93}$$

with the additional conditions:

$$c\boldsymbol{w}_{n;0} - \boldsymbol{v}_{;n} = 0, \tag{2.94}$$

$$c\,\boldsymbol{w}_{i;n} - \boldsymbol{v}_{;n} = 0. \tag{2.95}$$

Substitution of (2.91) into the last relations yields:

$$\boldsymbol{a}_{;n}\boldsymbol{v} + \boldsymbol{a}\boldsymbol{v}_{;n} - K\boldsymbol{v}_{;0} + c\big(\boldsymbol{a}^i{}_{;n}\boldsymbol{w}_i + \boldsymbol{a}^i\boldsymbol{w}_{i;n} - Kg^{ij}\boldsymbol{w}_{j;i}\big) = 0, \tag{2.96}$$

$$K\boldsymbol{v}_{;n} + \boldsymbol{a}_{;0}\boldsymbol{v} + c\boldsymbol{a}^j{}_{;0}\boldsymbol{w}_j + \boldsymbol{a}\boldsymbol{v}_{;0} + c\boldsymbol{a}^j\boldsymbol{w}_{j;0} = 0, \tag{2.97}$$

$$Kc\boldsymbol{w}_{i;n} + \boldsymbol{a}_{;i}\boldsymbol{v} + c\boldsymbol{a}^j{}_{;i}\boldsymbol{w}_j + \boldsymbol{a}\boldsymbol{v}_{;i} + c\boldsymbol{a}^j\boldsymbol{w}_{j;i} = 0. \tag{2.98}$$

On taking the scalar product of (2.97) by $\boldsymbol{a}$ and (2.98) by $\boldsymbol{a}^i$, we have:

$$K\boldsymbol{a}\boldsymbol{v}_{;n} + \boldsymbol{a}\boldsymbol{a}_{;0}\boldsymbol{v} + c\boldsymbol{a}\boldsymbol{a}^j{}_{;0}\boldsymbol{w}_j + \boldsymbol{a}^2\boldsymbol{v}_{;0} + c\boldsymbol{a}\boldsymbol{a}^j\boldsymbol{w}_{j;0} = 0, \tag{2.99}$$

$$Kc\boldsymbol{a}^i\boldsymbol{w}_{i;n} + \boldsymbol{a}^i\boldsymbol{a}_{;i}\boldsymbol{v} + c\boldsymbol{a}^i\boldsymbol{a}^j{}_{;i}\boldsymbol{w}_j + \boldsymbol{a}^i\boldsymbol{a}\boldsymbol{v}_{;i} + c\boldsymbol{a}^i\boldsymbol{a}^j\boldsymbol{w}_{j;i} = 0. \tag{2.100}$$

Since $\boldsymbol{v}, \boldsymbol{w}_i$ are partial derivatives of a field which is scalar, according to (2.40-b-c), it follows that:

$$(\boldsymbol{v} = \phi_{;0} \equiv \phi_{,0}, \quad \boldsymbol{w}_j = \phi_{;j} \equiv \phi_{,j}) \quad \Longrightarrow \quad (\boldsymbol{w}_{j;0} = \boldsymbol{v}_{;j}, \quad \boldsymbol{w}_{i;j} = \boldsymbol{w}_{j;i}).$$

Then the second derivatives being symmetric respect to indices, eqs (2.99), (2.100) become:

$$K\boldsymbol{a}\boldsymbol{v}_{;n}+\boldsymbol{a}\boldsymbol{a}_{;0}\boldsymbol{v}+c\boldsymbol{a}\boldsymbol{a}^j{}_{;0}\boldsymbol{w}_j+\boldsymbol{a}^2\boldsymbol{v}_{;0}+\tfrac{1}{2}c\big(\boldsymbol{a}\boldsymbol{a}^j+\boldsymbol{a}^j\boldsymbol{a}\big)\boldsymbol{w}_{j;0}=0, \quad (2.101)$$

$$\begin{aligned}&Kc\boldsymbol{a}^i\boldsymbol{w}_{i;n}+\boldsymbol{a}^i\boldsymbol{a}_{;i}\boldsymbol{v}+c\boldsymbol{a}^i\boldsymbol{a}^j{}_{;i}\boldsymbol{w}_j+\boldsymbol{a}^i\boldsymbol{a}\boldsymbol{v}_{;i}+\\&+\tfrac{1}{2}c\big(\boldsymbol{a}^i\boldsymbol{a}^j+\boldsymbol{a}^j\boldsymbol{a}^k\big)\boldsymbol{w}_{j;i}=0.\end{aligned} \quad (2.102)$$

Addition of (2.101), (2.102) to (2.96), multiplied by K, after some manipulations, leads to:

$$\begin{aligned}&\big(K\boldsymbol{a}_{;n}+\boldsymbol{a}\boldsymbol{a}_{;0}+\boldsymbol{a}^j\boldsymbol{a}_{;j}\big)\boldsymbol{v}+(\boldsymbol{a}^2-K^2\boldsymbol{I})\boldsymbol{v}_{;0}+\tfrac{1}{2}\big(\boldsymbol{a}^i\boldsymbol{a}+\boldsymbol{a}\boldsymbol{a}^i\big)\boldsymbol{v}_{;i}+\\&+c\big(K\boldsymbol{a}^j{}_{;n}+\boldsymbol{a}\boldsymbol{a}^j{}_{;0}+\boldsymbol{a}^i\boldsymbol{a}^j{}_{;i}\big)\boldsymbol{w}_j+\tfrac{1}{2}\big(\boldsymbol{a}\boldsymbol{a}^j+\boldsymbol{a}^j\boldsymbol{a}\big)\boldsymbol{w}_{j;0}+\\&+\tfrac{1}{2}c\big(\boldsymbol{a}^i\boldsymbol{a}^j+\boldsymbol{a}^j\boldsymbol{a}^i-2K^2g^{ij}\boldsymbol{I}\big)\boldsymbol{w}_{j;i}=0.\end{aligned} \quad (2.103)$$

The last condition (2.103) holds for any value of the field variables and their derivatives, if:

$$\boldsymbol{a}^2=K_2\boldsymbol{I}, \quad (2.104)$$

$$K\boldsymbol{a}_{;n}+\boldsymbol{a}\boldsymbol{a}_{;0}+\boldsymbol{a}^j\boldsymbol{a}_{;j}=0, \quad (2.105)$$

$$\boldsymbol{a}\boldsymbol{a}^j+\boldsymbol{a}^j\boldsymbol{a}=0, \quad (2.106)$$

$$\boldsymbol{a}^i\boldsymbol{a}^j+\boldsymbol{a}^j\boldsymbol{a}^i=2K^2g^{ij}\boldsymbol{I}. \quad (2.107)$$

Contraction of indices i, j in (2.107) yields:

$$\boldsymbol{a}^j\boldsymbol{a}_j=3K^2\boldsymbol{I}, \quad (2.108)$$

being $g^{ij}g_{ij}=3$. Now addition of (2.104) and (2.108) leads to:

$$\boldsymbol{a}^2+\boldsymbol{a}^j\boldsymbol{a}_j=4K^2\boldsymbol{I}. \quad (2.109)$$

Comparison of (2.109) and (2.92) implies, eventually:

$$\mathsf{a} = 4K^2\boldsymbol{\alpha}, \qquad \mathsf{a}_j = 4K^2\boldsymbol{\alpha}_j. \tag{2.110}$$

Now if we choose $K = \frac{1}{4}$, and introduce (2.110) into (2.107) we obtain:

$$\boldsymbol{\alpha}^i\boldsymbol{\alpha}^j + \boldsymbol{\alpha}^j\boldsymbol{\alpha}^i = 2g^{ij}\,\mathsf{I}. \tag{2.111}$$

Moreover, from (2.106) and (2.111) it follows that:

$$\boldsymbol{\alpha}_{;0} = 0, \qquad \boldsymbol{\alpha}_{;i} = 0, \qquad \boldsymbol{\alpha}^j{}_{;0} = 0, \qquad \boldsymbol{\alpha}^j{}_{;i} = 0. \tag{2.112}$$

In fact, we have *e.g.*:

$$\boldsymbol{\alpha}^1\boldsymbol{\alpha}^1 = g^{11}\,\mathsf{I} \quad \Longrightarrow \quad \boldsymbol{\alpha}^1\boldsymbol{\alpha}^1{}_{;i} = 0, \tag{2.113}$$

and being $\boldsymbol{\alpha}^1$ non-singular, it follows:

$$\boldsymbol{\alpha}^1{}_{;i} = 0. \tag{2.114}$$

The same happens for any other index value. It is easy to recognize in the previous relations a general relativistic extension of the same anti-commutation rule holding for the Dirac matrices.

So the equation (2.57-a) or (2.86), thanks to (2.89), finally results to be equivalent to a field equation of the form:

$$\boldsymbol{\alpha}\boldsymbol{w}_{n;0} + \boldsymbol{\alpha}^i\boldsymbol{w}_{n;i} + i\frac{mc}{\hbar}\boldsymbol{w}_n = 0, \tag{2.115}$$

which is recognized to be the Dirac equation:

$$\frac{\partial\psi}{\partial t} + c\,\boldsymbol{\alpha}^i\psi_{;i} + i\frac{mc^2}{\hbar}\beta\psi = 0, \tag{2.116}$$

when it is set:

$$\boldsymbol{w}_n = \beta\psi, \qquad \boldsymbol{\alpha} = \beta^{-1}. \tag{2.117}$$

2.4 Operators

2.4.1 Correspondence Principle

As we have seen until now the analytical unification between waves and particles we have performed, has led us to conceive a front wave as a family or set of the actual positions of particles of the same rest mass m.

The introduction of a suitable field dynamics shows that each wave front, in its turn, is required to be a solution of the equations governing some field $\boldsymbol{\phi}$ (or ψ). Therefore the field appears as a family of all the waves allowed as solutions of the field equations. In some sense we can conceive the field as a *family of families of particles*. Now the correspondence principle of *quantum mechanics*:

$$p_\alpha \longrightarrow \mathsf{P}_\alpha = -i\hbar \frac{\partial}{\partial x^\alpha}, \tag{2.118}$$

appears to replace the momentum of an individual particle of rest mass m with an *operator* involving the momenta of the all the particles of the family of families governed by the field.

Then the Klein-Gordon equation, when written in terms of the operator P_α, *i.e.*:

$$g^{\alpha\beta}\mathsf{P}_\alpha \mathsf{P}_\beta \boldsymbol{\phi} = m^2c^2\boldsymbol{\phi}, \tag{2.119}$$

looks like a natural generalization, to a family of families of particles, of the on-shell condition:

$$g^{\alpha\beta}p_\alpha p_\beta = m^2c^2, \tag{2.120}$$

holding for a single particle of rest mass m. The Hamiltonian density of the field, given by (2.69), provides also information about the number n of particles (*i.e.,* quanta of energy $\hbar\omega$) which are present in the volume V of the region filled by the field, resulting:

$$\mathcal{H} = \frac{n\hbar\omega}{V}. \tag{2.121}$$

Comparison of (2.121) and (2.69) yields:

$$\frac{n\hbar\omega}{V} = \frac{\varkappa\sqrt{|g|}\,\omega^2}{c^2}\,|\boldsymbol{\phi}|^2. \tag{2.122}$$

So the probability density of finding a particle in the unit volume of the region occupied by the field $\boldsymbol{\phi}$ results:

$$\varpi \equiv \frac{n}{V} = \frac{\varkappa\sqrt{|g|}\,\omega}{\hbar c^2}\,|\boldsymbol{\phi}|^2. \tag{2.123}$$

A suitable normalization of the field $\boldsymbol{\phi}$ allows to interpret $|\boldsymbol{\phi}|^2$ just as equal to the probability density of particle presence. The latter result which is usually *postulated* in quantum mechanics, here results naturally from energetic considerations about the field.

2.4.2 Uncertainty Principle

Introduction of operators, through the correspondence principle, leads also to the *uncertainty principle:*

$$\Delta A \Delta B \geq \tfrac{1}{2}\Big|[\mathsf{A}, \mathsf{B}]\Big|, \tag{2.124}$$

where:

$$[\mathsf{A}, \mathsf{B}] = \mathsf{A}\mathsf{B} - \mathsf{B}\mathsf{A}, \tag{2.125}$$

because of non-commutative algebra of any pair of conjugate operators A, B. In particular when:

$$\mathsf{A} = \mathsf{P} \equiv -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}, \qquad \mathsf{B} = \mathsf{X} \equiv x\,\mathsf{I}, \tag{2.126}$$

it results:

$$\Delta P \Delta X \geq \tfrac{1}{2}\hbar, \tag{2.127}$$

being:

$$[\mathsf{X}, \mathsf{P}] = i\hbar \mathsf{I}. \tag{2.128}$$

The result is not surprising since the energy of the field $\boldsymbol{\phi}$ is not sharply localized on a wave front of equation $\varphi = 0$, being distributed along the whole extension Δx of a wave pulse which depends on the superposition of the Fourier components required to localize the pulse extension. The frequencies and the wave numbers of such Fourier components stretching along the ranges $\Delta\omega, \Delta k$ determine, in the meanwhile an uncertainty of the energy and momenta $\Delta E = \hbar\Delta\omega,\ \Delta p = \hbar\Delta k$ of the wave-particle.

2.5 Conclusion

In this chapter we have formulated, in a more rigorous way, the ideas suggested heuristically in the first chapter and we have developed non-trivial consequences in order to a possible conceptual unification of waves and families of particles.

Moreover we have investigated some requested conditions for the possible fields which may govern the dynamics of wave-particles. Remarkably we have been quite naturally driven to the Klein-Gordon and Dirac equations as field equations, the solutions of which provide waves and particle families. The results have been obtained with the aid of a simplifying assumption that the metric is synchronous. In the next chapter such special condition will be removed and a quite general theory will be developed.

Chapter 3

Wave-Particles in V^n (Covariant Formulation)

Abstract
The present chapter generalizes in a fully covariant formulation all the results presented in the previous ones. In particular wave and particle mechanics, governed by the same *p.d.e.* will be treated within a n-*dimensional* space-time V^n with no assumptions on the co-ordinate system. A covariant Lagrangian density and the related energy-momentum tensor of a field which is responsible of wave-particles living on light-like paths in a higher dimensional space-time is obtained. Covariant Klein-Gordon and Dirac equations will be naturally deduced.

3.1 Introduction

In this chapter we present a generalization of the results, previously obtained under the restrictive assumption of the *synchronous gauge* condition. Here we will remove the time synchronization co-ordinate condition on the metric choice (*i.e.,* $g_{00} = 1, g_{0\bar{i}} = 0$) and present an explicitly covariant formulation of the proposed wave-particle unification approach, with no gauge assumptions. For the sake of facility for the reader, we will organize the sections following the same scheme and steps as in the previous chapter (see [29]). As we will see the results assume a more elegant and compact form when expressed in the explicitly covariant formalism, as it is usual in relativity.

3.2 Waves and Particle Dynamics

Let us consider a n-*dimensional* differentiable manifold V^n, representing a space-time endowed with a symmetric metric $\boldsymbol{g}$ of signature $(+, -, \cdots, -)$ and a torsionless connection $\boldsymbol{\Gamma}$. On V^n we represent any system of curvilinear co-ordinates with $x^{\bar{\alpha}}$ ($\bar{\alpha} = 0, \bar{i}$, with $\bar{i} = 1, 2, \cdots, n-1$). The physically observable space-time is then described by a 4-*dimensional* sub-manifold V^4 on which the observable co-ordinates are labelled by the indices $\alpha = 0, i$ with $i = 1, 2, 3$, while we will label the remaining co-ordinates $x^4, x^5, \cdots, x^{n-1}$ by underlined Latin indices $\underline{i} = 4, 5, \cdots, n-1$. Let us now consider any differentiable real valued scalar function $\varphi(x^{\bar{\alpha}})$, which we can always assume to be dimensionless. Then we can interpret the equation:

$$\varphi(x^{\bar{\alpha}}) = 0, \tag{3.1}$$

as the space-time world-sheet of a wave-front traveling across the $(n-1)$-*dimensional* space. (Some notes on *covariant* non-linear wave propagation theory have been proposed in Appendix B to facilitate the reader). Of course the function φ is determined by (3.1) except for an arbitrary non vanishing scalar factor, the choice of which is related to particle mass constance condition. The space-time path (*world-line*) of each point of the wave-sheet can be described by the parametric equations:

$$x^{\bar{\alpha}} \equiv x^{\bar{\alpha}}(\sigma), \tag{3.2}$$

where σ is a suitable evolution parameter. Substituting (3.2) into (3.1) and differentiating φ with respect to σ we obtain the differential equation governing the wave space-time geometry:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\sigma} \equiv V^{\bar{\alpha}} \frac{\partial \varphi}{\partial x^{\bar{\alpha}}} = 0,$$

where:

$$V^{\bar{\alpha}} = \frac{\mathrm{d}}{\mathrm{d}\sigma} x^{\bar{\alpha}}(\sigma), \tag{3.3}$$

is a tangent vector to the path at the point of co-ordinates $x^{\bar{\alpha}}$. We precise that σ can be chosen equal to the proper time or proper length (which are Riemannian invariant scalars) only if the trajectory is time-like, while a light-like trajectory can be treated only as a limiting case. Anyway the scalar condition:

$$V^{\bar{\alpha}} \frac{\partial \varphi}{\partial x^{\bar{\alpha}}} = 0, \qquad V^{\bar{\alpha}} V_{\bar{\alpha}} \leq 0, \tag{3.4}$$

is always preserved and the light speed is included as a limiting case when the vector of components $V^{\bar{\alpha}}$ has null length. The first step of our work consists in asking whether and at which conditions the wave ray equation (3.4) can be interpreted also as a covariant Hamilton-Jacobi equation:

$$g^{\bar{\alpha}\bar{\beta}} \frac{\partial S}{\partial x^{\bar{\alpha}}} \frac{\partial S}{\partial x^{\bar{\beta}}} - m^2 c^2 = 0, \tag{3.5}$$

governing the motion of some family of particles of rest mass m.

In order to make such interpretation possible, we may think of the Hamilton generating function S of the particle dynamics as proportional to the function φ.

So we set the relation:

$$S = \alpha\varphi, \tag{3.6}$$

where α is a suitable dimensional (positive) constant which we have already recognized to be equal to Planck reduced constant $\hbar$ for compatibility with a physical interpretation of the theory (see §2.2.2 in chapter 2).

Now we can identify (3.4-a) with (3.5) provided that the following conditions are fulfilled:

$$V^{\bar{\alpha}} = g^{\bar{\alpha}\bar{\beta}} \frac{\partial \varphi}{\partial x^{\bar{\beta}}}, \qquad m = 0. \tag{3.7}$$

It is immediate that compatibility requires:

$$V^{\bar{\alpha}} V_{\bar{\alpha}} = 0, \tag{3.8}$$

i.e., the particle must travel at the speed of light c, having zero rest mass.

According to the covariant theory of Hamilton-Jacobi, the canonical momentum of the particle is given by:

$$p_{\bar{\alpha}} = \frac{\partial S}{\partial x^{\bar{\alpha}}} \ , \tag{3.9}$$

and the Hamiltonian assumes the form:

$$H = \tfrac{1}{2\mathcal{K}} g^{\bar{\alpha}\bar{\beta}} \frac{\partial S}{\partial x^{\bar{\alpha}}} \frac{\partial S}{\partial x^{\bar{\beta}}}, \qquad \mathcal{K} = \textit{constant.} \tag{3.10}$$

(On the covariant formulation of the theory of Hamilton-Jacobi, see Appendix C and also [17]).

3.2.1 Covariant Hamilton Equations

Then the Hamilton equations:

$$\frac{\mathrm{d}\, p_{\bar{\alpha}}}{\mathrm{d}\sigma} = -\frac{\partial H}{\partial x^{\bar{\alpha}}}, \qquad \frac{\mathrm{d}\, x^{\bar{\alpha}}}{\mathrm{d}\sigma} = \frac{\partial H}{\partial p_{\bar{\alpha}}}, \tag{3.11}$$

become explicitly:

$$\frac{\mathrm{d}\, p_{\bar{\alpha}}}{\mathrm{d}\sigma} = -\tfrac{1}{2\mathcal{K}}\, p_{\bar{\mu}}\, p_{\bar{\nu}}\, g^{\bar{\mu}\bar{\nu}}{}_{,\bar{\alpha}}, \tag{3.12}$$

$$\frac{\mathrm{d}\, x^{\bar{\alpha}}}{\mathrm{d}\sigma} = \tfrac{1}{\mathcal{K}}\, p^{\bar{\alpha}}. \tag{3.13}$$

One realizes that (3.12) is equivalent to the geodesic condition:

$$\frac{\mathrm{d}\, p_{\bar{\alpha}}}{\mathrm{d}\sigma} - \varGamma^{\bar{\mu}}_{\bar{\alpha}\bar{\nu}}\, p_{\bar{\mu}}\, \frac{\mathrm{d}\, x^{\bar{\nu}}}{\mathrm{d}\sigma} = 0, \tag{3.14}$$

which, thanks to (3.13), becomes:

$$\frac{\mathrm{d}\, p_{\bar{\alpha}}}{\mathrm{d}\sigma} - \tfrac{1}{\mathcal{K}}\varGamma^{\bar{\mu}}_{\bar{\alpha}\bar{\nu}}\, p_{\bar{\mu}}\, p^{\bar{\nu}} = 0, \tag{3.15}$$

which leads to (3.12). In fact:

$$\tfrac{1}{2} g^{\bar{\mu}\bar{\nu}}{}_{,\,\bar{\alpha}}\; p_{\bar{\mu}}\, p_{\bar{\nu}} \equiv -\varGamma^{\bar{\mu}}_{\bar{\nu}\bar{\alpha}}\, p_{\bar{\mu}}\, p^{\bar{\nu}}, \tag{3.16}$$

being:

$$g^{\bar{\mu}\bar{\nu}}{}_{;\,\bar{\alpha}} \equiv g^{\bar{\mu}\bar{\nu}}{}_{,\bar{\alpha}} + \varGamma^{\bar{\mu}}_{\bar{\alpha}\bar{\rho}}\, g^{\bar{\rho}\bar{\nu}} + \varGamma^{\nu}_{\bar{\alpha}\bar{\rho}}\, g^{\bar{\mu}\bar{\rho}} = 0, \tag{3.17}$$

thanks to the metricity condition.

Therefore we may conclude that the differential equation governing wave kinematics may be identified with the Hamilton-Jacobi equation governing the motion of a massless particle traveling at the speed of light across the $(n-1)$-*dimensional* space, provided that the wave itself travels at the speed of light. Now we can conveniently split the n components of the vectors $(p_{\bar{\alpha}})$ and $(x^{\bar{\alpha}})$ into the components onto the physical sub-space-time p_α, x^α and the *extra* space components $p_{\underline{i}}$, $x^{\underline{i}}$.

We obtain the following decompositions:

$$\begin{aligned} \frac{\mathrm{d}\, p_\alpha}{\mathrm{d}\sigma} &= -\tfrac{1}{2\mathcal{K}}\left(g^{\mu\nu}{}_{,\alpha}p_\mu p_\nu + g^{\underline{j}\underline{k}}{}_{,\alpha}p_{\underline{j}}p_{\underline{k}}\right) - \tfrac{1}{\mathcal{K}}g^{\mu\underline{k}}{}_{,\alpha}p_\mu p_{\underline{k}}, \\ \frac{\mathrm{d}\, x^\alpha}{\mathrm{d}\sigma} &= \tfrac{1}{\mathcal{K}}p^\alpha, \end{aligned} \tag{3.18}$$

$$\begin{aligned} \frac{\mathrm{d}\, p_{\underline{i}}}{\mathrm{d}\sigma} &= -\tfrac{1}{2\mathcal{K}}\left(g^{\mu\nu}{}_{,\underline{i}}p_\mu p_\nu + g^{\underline{j}\underline{k}}{}_{,\underline{i}}p_{\underline{j}}p_{\underline{k}}\right) - \tfrac{1}{\mathcal{K}}g^{\mu\underline{k}}{}_{,\underline{i}}p_j p_{\underline{k}}, \\ \frac{\mathrm{d}\, x^{\underline{i}}}{\mathrm{d}\sigma} &= \tfrac{1}{\mathcal{K}}p^{\underline{i}}. \end{aligned} \tag{3.19}$$

We can interpret as a particle rest mass the term:

$$m = \tfrac{1}{c}\sqrt{-g^{\underline{j}\underline{k}}p_{\underline{j}}p_{\underline{k}} - 2g^{\alpha\underline{k}}p_\alpha p_{\underline{k}}}, \tag{3.20}$$

being:

$$p^{\bar{\alpha}}p_{\bar{\alpha}} \equiv g^{\alpha\beta}p_\alpha p_\beta + g^{\underline{j}\underline{k}}p_{\underline{j}}p_{\underline{k}} + 2g^{\alpha\underline{k}}p_\alpha p_{\underline{k}} = 0. \tag{3.21}$$

The rest mass m results to be constant if we assume:

$$-g^{\underline{j}\underline{k}}p_{\underline{j}}p_{\underline{k}} - 2g^{\alpha\underline{k}}p_\alpha p_{\underline{k}} = \textit{non-negative constant}. \tag{3.22}$$

Such a choice is always, at least mathematically, possible since $p_{\bar{\alpha}} = \alpha\varphi_{,\bar{\alpha}}$ and φ is defined except for a non vanishing factor according to (3.1). More investigations remain to be exploited in order to understand a physical “mechanism” ensuring such a constance of mass (see chapter 6 in Part II).

Then we gain, in the physical sub-space-time, the momentum of a massive particle, resulting:

$$g^{\alpha\beta}p_\alpha p_\beta = m^2c^2. \tag{3.23}$$

We point out that when $g^{\alpha\beta}$ depends only on the observable co-ordinates x^{μ}, (3.18-a) becomes the equation of the geodesic path run by a particle of rest mass m crossing the physical space-time V^4 in presence of a gravitational field, while the fields described by $g_{\underline{j}\,\underline{k}}, g_{\mu\underline{k}}$, may be interpreted as non-gravitational fields in more than four space-time dimensions like a Kaluza-Klein type theory or some alternative to this latter.

3.2.2 Covariant De Broglie-Einstein-Planck Relation

The Hamilton generating function S, proportional to the function φ, can easily be evaluated by integration of (3.4-a), obtaining:

$$\varphi = f(V_{\bar{\alpha}}\, x^{\bar{\alpha}}). \tag{3.24}$$

In fact it results:

$$\frac{\partial \varphi}{\partial x^{\bar{\alpha}}} = f' V_{\bar{\alpha}}, \qquad V_{\bar{\alpha}} = \tfrac{1}{\mathcal{K}} p_{\bar{\alpha}},$$

$p_{\bar{\alpha}}, x^{\bar{\alpha}}$ being independent variables and *prime* denoting the derivative respect to the function argument.

In detail we have: $\varphi_{,\bar{\alpha}} \equiv f' V_{\bar{\alpha}} + f' V_{\bar{\beta};\bar{\alpha}} x^{\bar{\beta}}$, being $V_{\bar{\beta}} V^{\bar{\beta}} = 0$.

Since the wave propagates along a path of equation $x^{\bar{\alpha}} = V^{\bar{\alpha}}\sigma$, it follows:

$V_{\bar{\beta};\bar{\alpha}} x^{\bar{\alpha}} \equiv V_{\bar{\beta};\bar{\alpha}} V^{\bar{\beta}} \sigma = 0$, since: $V_{\bar{\beta};\bar{\alpha}} V^{\bar{\beta}} = 0$.

It follows:

$$V^{\bar{\alpha}} \frac{\partial \varphi}{\partial x^{\bar{\alpha}}} \equiv V^{\bar{\alpha}} V_{\bar{\alpha}} f' = 0,$$

the norm of $V^{\bar{\alpha}}$ being null, since the wave-particle is traveling at the speed of light in V^n. Then the canonical momentum, evaluated during wave-particle motion, becomes:

$$p_{\bar{\alpha}} = \alpha \varphi' V_{\bar{\alpha}}, \tag{3.25}$$

thanks to (3.6) and (3.9). Now we can introduce the following quantity related to wave propagation:

$$k_{\bar{\alpha}} = \varphi' V_{\bar{\alpha}}. \tag{3.26}$$

On identifying the constant α with Planck constant the covariant relation including both De Broglie and Einstein-Planck relations arises naturally, *i.e.*:

$$p_{\bar{\alpha}} = \hbar k_{\bar{\alpha}}. \tag{3.27}$$

In the special case of a periodic wave it is immediate to recognize the quantities $k_{\bar{\alpha}}$ as the usual components of the n-*dimensional* wave number vector. In general it is convenient to choose $k_{\bar{\alpha}} x^{\bar{\alpha}}$ as argument of wave solutions, since it results simply:

$$\varphi = k_{\bar{\alpha}} x^{\bar{\alpha}}, \tag{3.28}$$

because of compatibility with (3.27).

3.3 Wave Dynamics and Field Equations

The problem of wave dynamics consists in determining a class of *fields* governed by *field equations* which yield the wave solutions equivalent to the solutions to the equation of Hamilton-Jacobi for particles, as examined in the previous sections.

Of course the main condition required to the system of field equations is that it must provide waves traveling with the speed of light c across an

$(n-1)$-*dimensional* space, but that condition alone is too loose to characterize a physically meaningful class of field equations. So we need some other reasonable assumptions.

Therefore we will require the following ones:

1. The system of field equations is required to be *Lagrangian.*
2. The production term of the equations governing $\boldsymbol{\phi}$ must be zero, or at least null in correspondence to the simple wave solutions. This second assumption is required in order to provide regular solutions to the system as simple waves, ensuring that the particles travel along the rays at characteristic speeds. In principle also *discontinuity waves* (see [7]) could be considered, but if we want to be able to compare the results with quantum mechanical ones we need regular solutions which can be expanded into Fourier series.
3. The normal speed of all simple waves must be equal to the speed of light c. This one is the true kinematic condition we have previously examined.

3.3.1 The Lagrangian System

Let us consider a candidate field $\boldsymbol{\phi}$, which in general may be a complex column vector belonging to an N-dimensional Euclidean complex space, and is assumed to be a set of regular scalar functions of $x^{\bar{\alpha}}$, which is invariant respect to any regular co-ordinate transformation. And let:

$$\mathcal{L} \equiv \sqrt{|g|}\, \ell\big(\boldsymbol{w}_{\bar{\alpha}}, \boldsymbol{w}^{+}_{\bar{\alpha}}\big), \qquad \boldsymbol{w}_{\bar{\alpha}} = \boldsymbol{\phi}_{,\bar{\alpha}}, \tag{3.29}$$

be a Lagrangian density governing the field dynamics. Such a Lagrangian density is supposed to depend only on $\boldsymbol{w}_{\bar{\alpha}}$, in order to fulfill assumption(2).

Possible production terms depending explicitly on $\boldsymbol{\phi}$ will be assumed to be null or negligible. In the present section, for simplicity, they will simply be dropped. Then the system of first order field equations results:

$$\left(\sqrt{|g|}\,\frac{\partial \ell}{\partial \boldsymbol{w}^{+}_{\bar{\alpha}}}\right)_{,\alpha} = 0, \tag{3.30}$$

$$\boldsymbol{w}_{\bar{\beta},\bar{\alpha}} - \boldsymbol{w}_{\bar{\alpha},\bar{\beta}} = 0, \tag{3.31}$$

together with its complex conjugate. The second equation arises from Schwarz compatibility condition which holds for each scalar component of $\boldsymbol{\phi}$.

On applying the correspondence rule:

$$(\,\cdot\,)_{,\alpha} \longrightarrow \varphi_{,\bar{\alpha}}\,(\,\cdot\,)', \tag{3.32}$$

prime denoting here differentiation respect to φ, we obtain, according to wave propagation theory, the algebraic system for the simple waves:

$$\varphi_{,\bar{\alpha}}\left(\sqrt{|g|}\,\frac{\partial \ell}{\partial \boldsymbol{w}^{+}_{\bar{\alpha}}}\right)' = 0, \tag{3.33}$$

$$\boldsymbol{w}'_{\bar{\beta}}\,\varphi_{,\bar{\alpha}} - \boldsymbol{w}'_{\bar{\alpha}}\,\varphi_{,\bar{\beta}} = 0, \tag{3.34}$$

in which the unknown field variables $\boldsymbol{w}_{\bar{\alpha}}$ are functions of φ. From (3.33) we have soon:

$$\varphi_{,\bar{\alpha}}\,\varphi_{,\bar{\beta}}\,\frac{\partial^2 \ell}{\partial \boldsymbol{w}^{+}_{\bar{\alpha}}\,\partial \boldsymbol{w}_{\bar{\alpha}}}\,\boldsymbol{\phi}' = 0. \tag{3.35}$$

Since the Lagrangian density is to be independent on the wave-sheet geometry, *i.e.,* on $\varphi_{,\bar{\alpha}}$, and the only admissible simple waves are supposed to travel at the speed of light, so that we have:

$$g^{\bar{\alpha}\bar{\beta}}\,\varphi_{,\bar{\alpha}}\,\varphi_{,\bar{\beta}} = 0,$$

the following conditions must hold:

$$\frac{\partial^2 \ell}{\partial \boldsymbol{w}^{+}_{\bar{\alpha}}\,\partial \boldsymbol{w}_{\bar{\beta}}} = g^{\bar{\alpha}\bar{\beta}}\,\boldsymbol{a}, \tag{3.36}$$

where a is some non singular Hermitean matrix, independent of the fields. From those conditions the form of the Lagrangian density is determined as:

$$\mathcal{L} = \tfrac{1}{2}\sqrt{|g|}\; g^{\bar{\alpha}\bar{\beta}}\, \boldsymbol{w}^{+}_{\bar{\alpha}}\, \mathsf{a}\boldsymbol{w}_{\bar{\beta}}\,. \tag{3.37}$$

Then the Euler-Lagrange equations are given by:

$$\left(\sqrt{|g|}\; g^{\bar{\alpha}\bar{\beta}}\, \mathsf{a}\boldsymbol{w}_{\bar{\beta}}\right)_{,\alpha} = 0 \qquad \Longleftrightarrow \qquad (g^{\bar{\alpha}\bar{\beta}}\, \mathsf{a}\boldsymbol{w}_{\bar{\beta}})_{;\bar{\alpha}} = 0. \tag{3.38}$$

We observe that a, beside being independent of the fields $\boldsymbol{w}_{\bar{\alpha}}$, must be independent also of $x^{\bar{\alpha}}$, otherwise a non vanishing production term would arise into the field equations.

Then, since a is a non singular matrix, taking into account the metricity condition $g^{\bar{\alpha}\bar{\beta}}_{\;\;;\bar{\mu}} = 0$, the field equations result simply:

$$g^{\bar{\alpha}\bar{\beta}}\, \boldsymbol{w}_{\bar{\beta}\,;\bar{\alpha}} = 0, \qquad \boldsymbol{w}_{\bar{\beta}\,;\bar{\alpha}} - \boldsymbol{w}_{\bar{\alpha}\,;\bar{\beta}} = 0. \tag{3.39}$$

Since the coefficient matrix a disappears form the equations its role becomes irrelevant in the Lagrangian. So it is not a restriction to choose:

$$\mathsf{a} = \varkappa \mathsf{I}, \tag{3.40}$$

where $\varkappa$ is a suitable dimensional constant and I is the identity matrix in the space of the field $\boldsymbol{\phi}$. Then the scalar Lagrangian density becomes:

$$\ell = \tfrac{1}{2}\varkappa\, g^{\bar{\alpha}\bar{\beta}}\, \boldsymbol{w}^{+}_{\bar{\alpha}}\, \boldsymbol{w}_{\bar{\beta}}\,. \tag{3.41}$$

We point out that the algebraic system which determines the simple waves, arising from (3.39), through direct substitution of $\boldsymbol{w}_{\bar{\alpha}} = \boldsymbol{\phi}'\varphi_{\,;\bar{\alpha}}$, yields:

$$g^{\bar{\alpha}\bar{\beta}}\, \varphi_{\,;\bar{\alpha}}\, \varphi_{\,;\bar{\beta}}\, \boldsymbol{\phi}'^{2} = 0. \tag{3.42}$$

Then for a non trivial solution ($\boldsymbol{\phi}' \neq 0$) we have a wave traveling at the speed of light, as expected, resulting just:

$$g^{\bar{\alpha}\bar{\beta}}\varphi_{;\bar{\alpha}}\varphi_{;\bar{\beta}} = 0. \tag{3.43}$$

3.3.2 The Energy-Momentum Tensor

Starting from the Lagrangian density of the field $\boldsymbol{\phi}$, given by (3.37), (3.41) we can evaluate the energy-momentum tensor, defined by:

$$\tfrac{1}{2}\sqrt{|g|}\,T_{\bar{\mu}\bar{\nu}} = \frac{\partial\mathcal{L}}{\partial g_{\bar{\mu}\bar{\nu}}} - \Big(\frac{\partial\mathcal{L}}{\partial g_{\mu\nu,\alpha}}\Big)_{,\alpha}, \tag{3.44}$$

obtaining:

$$T_{\bar{\mu}\bar{\nu}} = \varkappa \boldsymbol{w}^+_{\bar{\mu}}\boldsymbol{w}_{\bar{\nu}} + \tfrac{1}{2}\varkappa\frac{g}{|g|}g^{\bar{\alpha}\bar{\beta}}\boldsymbol{w}^+_{\bar{\alpha}}\boldsymbol{w}_{\bar{\beta}}\,g_{\bar{\mu}\bar{\nu}}\,. \tag{3.45}$$

From which we have the filed Hamiltonian density:

$$\mathcal{H} \equiv T_{00} = \varkappa\Big(1 + \tfrac{1}{2}\frac{g}{|g|}\Big)\boldsymbol{w}^+_0\boldsymbol{w}_0 + \varkappa\frac{g}{|g|}g^{\underline{j}\bar{\beta}}\boldsymbol{w}^+_{\underline{j}}\boldsymbol{w}_{\bar{\beta}}\,g_{00}, \tag{3.46}$$

which generalizes (2.65) which was obtained in the special case of a synchronous frame (see, §2.3.2 in chapter 2).

3.3.3 The Klein-Gordon Equation

Let us now study in more detail the system of field equations (3.39).

Second Order Formulation

First of all we point out that we can always replace the fields $\boldsymbol{w}_{\bar{\alpha}}$, with their original definitions in terms of derivatives of the field $\boldsymbol{\phi}$, according to (3.29-b), obtaining a system of N second order equations:

$$g^{\bar{\alpha}\bar{\beta}}\boldsymbol{\phi}_{;\bar{\alpha};\bar{\beta}} = 0, \tag{3.47}$$

instead of the original system of $2N$ first order equations (3.39). The latter generalized D'Alembert equation governing the propagation of the field $\boldsymbol{\phi}$ across the $(n-1)$-*dimensional* space, is equivalent to a Klein-Gordon equation for the propagation of the same field across the physical 3-*dimensional* space. In fact the solutions for the field $\boldsymbol{\phi}$, being simple waves, are composite functions through φ of the argument:

$$p_{\bar{\alpha}}\, x^{\bar{\alpha}} = \hbar k_{\bar{\alpha}}\, x^{\bar{\alpha}} \equiv \hbar \varphi' V_{\bar{\alpha}}\, x^{\bar{\alpha}}. \tag{3.48}$$

Therefore we can consider the field $\boldsymbol{\phi}$ as a function of the form:

$$\boldsymbol{\phi} \equiv \boldsymbol{\phi}(p_\alpha x^\alpha + p_{\underline{i}} x^{\underline{i}}). \tag{3.49}$$

Without affecting the four physical co-ordinates x^α, it is always possible to choose the *extra* space co-ordinates $x^{\underline{i}}$ in such a way that:

$$x^n = x^{\underline{i}} N_{\underline{i}}, \qquad g^{nn} \equiv g^{\underline{j}\underline{k}} N_{\underline{j}} N_{\underline{k}} = -1, \tag{3.50}$$

$$g^{\underline{i}j} = g^{j\underline{i}} = 0, \tag{3.51}$$

being:

$$p_{\underline{i}} = -mcN_{\underline{i}}, \qquad N_{\underline{i}} = \frac{p_{\underline{i}}}{\sqrt{-g^{\underline{j}\underline{k}} p_{\underline{j}} p_{\underline{k}}}}. \tag{3.52}$$

Then the simple wave live in a five dimensional *sub*-space-time, and:

$$\boldsymbol{\phi} \equiv \boldsymbol{\phi}\big(p_\alpha x^\alpha - mcx^n\big). \tag{3.53}$$

If we require the additional assumption (usual in quantum mechanics) that $\boldsymbol{\phi}$ belongs to the Hilbert space L^2, so that it can be expanded into Fourier series, we can write:

$$\boldsymbol{\phi} = \sum_{r=-\infty}^{+\infty} \boldsymbol{c}_r\, e^{-\frac{i}{\hbar}(p_{r\alpha} x^\alpha - mcx^n)} \equiv e^{\frac{i}{\hbar} mcx^n} \sum_{r=-\infty}^{+\infty} \boldsymbol{c}_r\, e^{-\frac{i}{\hbar}(p_{r\alpha} x^\alpha)}. \tag{3.54}$$

Then we are able to evaluate the Laplacian:

$$g^{\underline{i}\underline{j}}\, \boldsymbol{\phi}_{;\underline{i};\underline{j}} \equiv g^{nn}\, \boldsymbol{\phi}_{;n;n} \equiv \frac{m^2 c^2}{\hbar^2} \boldsymbol{\phi}.$$

The field equation (3.47) leads to the Klein-Gordon equation in generalized co-ordinates:

$$g^{\alpha\beta}\boldsymbol{\phi}_{;\alpha;\beta} + \frac{m^2c^2}{\hbar^2}\boldsymbol{\phi} = 0 \qquad \Longleftrightarrow \qquad \Box\boldsymbol{\phi} = \frac{m^2c^2}{\hbar^2}\boldsymbol{\phi}. \tag{3.55}$$

First Order Formulation

A further relevant consideration arises on evaluating the divergence of $\boldsymbol{w}_{\bar{\alpha}}$ in terms of the field variables $\boldsymbol{w}_{\alpha}, \boldsymbol{w}_{\underline{i}}$. Taking into account the previous results we have:

$$\boldsymbol{w}_{\underline{j}} = \boldsymbol{w}_n \delta_{\underline{j}n}, \qquad \boldsymbol{w}_{\underline{j};n} = \frac{imc}{\hbar}\boldsymbol{w}_n \delta_{\underline{j}n}. \tag{3.56}$$

It follows into the system (3.39):

$$g^{\alpha\beta}\boldsymbol{w}_{\beta;\alpha} + \frac{imc}{\hbar}\boldsymbol{w}_n = 0, \tag{3.57}$$

$$\boldsymbol{w}_{\alpha;n} - \boldsymbol{w}_{n;\alpha} = 0. \tag{3.58}$$

We observe that we need at least *one extra space dimension* to introduce the rest mass. Then at least five dimensions are required if the particle travels across a gravitational field. A result that suggests that a suitable Kaluza-Klein type theory could be a candidate to govern the matter field $\boldsymbol{\phi}$ together with gravitational and electromagnetic fields. More than five space-time dimensions are required if other fundamental fields are present, like Yang-Mills fields, to describe also weak and strong interactions. Starting from the next chapter we will attack the problem of field unification. However we will see how the usual Kaluza-Klein approach proves inadequate and we will propose an alternative model, which results to be more natural both on a mathematical and a physical standpoint.

3.3.4 The Dirac Equation

In the N-*dimensional* linear space of the field variables $\boldsymbol{w}_{\bar{\alpha}}$ it is always possible to find four non singular matrices relating a component of the 4-*vector* $\boldsymbol{w}_\alpha$ with the vector $\boldsymbol{w}_n$. So we can introduce a vector of matrices γ_α such that:

$$\boldsymbol{w}_\alpha = -K\gamma_\alpha \boldsymbol{w}_n, \tag{3.59}$$

where K is a suitable constant to be determined. (The notation is motivated since such matrices will be shown to be just the covariant Dirac matrices). The previous relation implies also:

$$\gamma^\alpha \boldsymbol{w}_\alpha = -K\gamma^\alpha\gamma_\alpha \boldsymbol{w}_n, \qquad \gamma^\alpha = g^{\alpha\beta}\gamma_\alpha. \tag{3.60}$$

Assuming that $\gamma^\alpha\gamma_\alpha$ is non singular it is possible to introduce another vector of matrices a^α such that:

$$\boldsymbol{w}_n = -\frac{1}{K}\mathsf{a}^\alpha \boldsymbol{w}_\alpha, \tag{3.61}$$

Combining (3.61) with (3.59) we get:

$$\boldsymbol{w}_n = \mathsf{a}^\alpha\gamma_\alpha \boldsymbol{w}_n.$$

And therefore:

$$\mathsf{a}^\alpha\gamma_\alpha = \mathsf{I}. \tag{3.62}$$

Thanks to (3.61) and taking account of Schwarz condition (holding for the scalar components of the field ϕ), an equivalent system to the field equations (3.39) may be written only in terms of $\boldsymbol{w}_\alpha$:

$$\boldsymbol{a}^\alpha{}_{;n}\boldsymbol{w}_\alpha + \boldsymbol{a}^\alpha \boldsymbol{w}_{\alpha;n} - K\, g^{\alpha\beta}\boldsymbol{w}_{\beta;\alpha} = 0, \tag{3.63}$$

$$K\boldsymbol{w}_{\alpha;n} + \boldsymbol{a}^\beta{}_{;\alpha}\boldsymbol{w}_\beta + \boldsymbol{a}^\beta \boldsymbol{w}_{\beta;\alpha} = 0. \tag{3.64}$$

On taking the scalar product of (3.64) by $\boldsymbol{a}^\alpha$ we have:

$$K\,\boldsymbol{a}^\alpha \boldsymbol{w}_{\alpha;n} + \boldsymbol{a}^\alpha\,\boldsymbol{a}^\beta{}_{;\alpha}\boldsymbol{w}_\beta + \boldsymbol{a}^\alpha\,\boldsymbol{a}^\beta \boldsymbol{w}_{\beta;\alpha} = 0. \tag{3.65}$$

Since $\boldsymbol{w}_\beta$ is a gradient of a field which is scalar, according to (3.29-b), it follows that:

$$\boldsymbol{w}_\beta = \boldsymbol{\phi}_{;\beta} \equiv \boldsymbol{\phi}_{,\beta} \quad \Longrightarrow \quad \boldsymbol{w}_{\beta;\alpha} = \boldsymbol{w}_{\alpha;\beta}.$$

Then $\boldsymbol{w}_{\beta;\alpha}$ being symmetric respect to α, β, (3.65) becomes:

$$K\,\boldsymbol{a}^\alpha \boldsymbol{w}_{\alpha;n} + \boldsymbol{a}^\alpha\,\boldsymbol{a}^\beta{}_{;\alpha}\boldsymbol{w}_\beta + \tfrac{1}{2}(\boldsymbol{a}^\alpha \boldsymbol{a}^\beta + \boldsymbol{a}^\beta \boldsymbol{a}^\alpha)\boldsymbol{w}_{\beta;\alpha} = 0. \tag{3.66}$$

Taking the sum of (3.66) and (3.63), after multiplying the latter by K, after some manipulations, we eliminate $\boldsymbol{w}_{\alpha;n}$. Therefore we obtain:

$$\left(K\,\boldsymbol{a}^\beta{}_{;n} + \boldsymbol{a}^\alpha\,\boldsymbol{a}^\beta{}_{;\alpha}\right)\boldsymbol{w}_\beta + \tfrac{1}{2}\left(\boldsymbol{a}^\alpha \boldsymbol{a}^\beta + \boldsymbol{a}^\beta \boldsymbol{a}^\alpha - 2K^2 g^{\alpha\beta}\,\boldsymbol{I}\right)\boldsymbol{w}_{\beta;\alpha} = 0. \tag{3.67}$$

The last condition (3.67) holds for any value of the field $\boldsymbol{w}_\beta$ and its derivatives $\boldsymbol{w}_{\beta;\alpha}$ if and only if:

$$K\,\boldsymbol{a}^\beta{}_{;n} + \boldsymbol{a}^\alpha\,\boldsymbol{a}^\beta{}_{;\alpha} = 0, \tag{3.68}$$

$$\boldsymbol{a}^\alpha \boldsymbol{a}^\beta + \boldsymbol{a}^\beta \boldsymbol{a}^\alpha = 2K^2 g^{\alpha\beta}\,\boldsymbol{I}. \tag{3.69}$$

Contraction of indices α, β in (3.69) yields:

$$\boldsymbol{a}^\alpha\,\boldsymbol{a}_\alpha = 4K^2\,\boldsymbol{I}, \tag{3.70}$$

being $g^{\alpha\beta}g_{\alpha\beta} = 4$. A comparison between (3.62) and (3.70) implies now:

$$\mathsf{a}_\alpha = 4K^2\gamma_\alpha. \tag{3.71}$$

Now we choose $K = \frac{1}{4}$, and introduce (3.71) into (3.69) obtaining:

$$\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2g^{\alpha\beta}\mathit{I}. \tag{3.72}$$

Moreover, from the latter anti-commutation relations it follows that:

$$\gamma^\beta{}_{;\alpha} = 0. \tag{3.73}$$

In fact, we have *e.g.*:

$$\gamma^1\gamma^1 = g^{11}\mathit{I} \quad \Longrightarrow \quad \gamma^1\gamma^1{}_{;\alpha} = 0. \tag{3.74}$$

Since γ^1 is non-singular, it follows:

$$\gamma^1{}_{;\alpha} = 0. \tag{3.75}$$

The same happens for any index value. It is easy to recognize in the last relation a general relativistic extension of the same anti-commutation rule holding for the Dirac matrices.

So the equation (3.39-a) or (3.57), thanks to (3.59), finally results to be equivalent to a field equation of the form:

$$\gamma^\alpha \boldsymbol{w}_{n;\alpha} + i\frac{mc}{\hbar}\boldsymbol{w}_n = 0, \tag{3.76}$$

which is recognized to be the Dirac equation:

$$\gamma^\alpha\psi_{;\alpha} + i\frac{mc}{\hbar}\psi = 0, \tag{3.77}$$

when it is set:

$$\boldsymbol{w}_n = \psi. \tag{3.78}$$

3.4 Conclusion

In the present chapter we have developed the theory presented in the previous one, in a completely general formulation, with no conditions on the co-ordinate system.

Se we were able to provide a fully covariant formulation of the theory of wave and particle family unification.

The dynamics led us to covariant Klein-Gordon and Dirac field equations in a very natural and elegant way as field equations required to govern unified wave-partcles.

The problem of field unification involving all the known physical interactions (*gravitational, electro-magnetic, weak* and *strong*) in a conceptually unified field will be the subject of a theory proposed in the second part of this book.

Part II

Field Unification

The *Part II* of the book deals with the problem of field unification and examines the possible field theories candidate to govern the wave-particles dynamics and their interactions with fundamental fields. Cosmological application is also examined and a way to gravity quantization is conjectured.

Chapter 4

Kaluza-Klein Theories

Abstract
This chapter opens the second part of the book, which is devoted to investigate what kind of unified field dynamics may be adequate to provide unified wave-particle solutions, as conceived in the previous chapters. Kaluza-Klein type theories seem to be reasonable candidates to be considered at first. Here we limit ourselves to sketch a possible way to deal with the *dilaton* ϕ as a massive particle field. But, as we will see a similar approach will not prove to be satisfactory and successful.

4.1 Introduction

In part I we proposed a possible strategy of unification of the concepts of wave and particle (or better, family of particles) centered on the idea that a formally identical partial differential equation allows a two-fold physical interpretation:

1. Either as the evolution law of a *wave,* the front of which is traveling across the physical space;
2. Or as the Hamilton-Jacobi equation governing motion of a *family of particles*.

We have seen how such dual interpretation of the same equation has a physical meaning provided that both the wave-front and the particles travel at the speed of light c. Therefore the particles need necessarily to have zero rest mass. So, in order to preserve such wave-particle unification scheme, even when non-vanishing rest mass particles exist, as it happens within the real world, a space-time with more than four dimensions seems to be required.

A second step we developed in the previous chapters was concerned with the investigation of the properties which are required to some field so that the unified wave-particle are possibile solutions of the field equations. In other words we have tried to answer to the question: "Our unified wave-particles are waves and particles *of what*?"

We have seen that what we need is a field or a set of more fields governed by a D'Alembertian equation in higher dimensional space-time, which results to be equivalent either to a Klein-Gordon equation (to describe *bosons*) or a Dirac equation (to describe *fermions*) in four dimensional space-time.

In Part II of this book we intend to investigate which of the known physical fields can be actually involved with the unified wave-particles as solutions, within a space-time endowed with more than four dimensions. The results presented in the previous chapters seem to suggest to investigate first of all the opportunity of interpreting the scalar *dilaton* of Brans-Dicke theory (in presence of pure gravitation), or of Kaluza-Klein theory (in presence of electromagnetic field) as a matter field responsible of particle rest masses. (On Brans-Dicke and Kaluza-Klein theories one can see the original papers [3], [23] and the more recent review article [25]. Some authors suggested also that the *dilaton* might be the scalar field observed at Cern LHC in 2012. see, *e.g.,* [2], [11]).

So in the present chapter we will investigate how to include one scalar field ϕ, into the metric tensor of the manifold V^n following a Kaluza-Klein type scheme (see [28, 29] and also [25], [30]).

Here we limit ourselves to sketch a possible way to deal with *dilaton* ϕ as a massive particle field. But, as we will see a similar approach will not prove to be satisfactory. So, in the next chapter a more physically and mathematically intriguing approach to the problem will be exploited.

4.2 Wave-Particles as Solutions in a Brans-Dicke Theory

Let us start considering the simple case of one only scalar field ϕ involved as *dilaton* in a Brans-Dicke theory, *i.e.* in a purely gravitational field (absence of Abelian electromagnetic and non-Abelian Yang-Mills fields).

So we suppose to work in a 5-*dimensional* space-time V^5 on which we represent the metric tensor and its inverse as:

$$(g_{\bar{\mu}\bar{\nu}}) \equiv \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & -\phi^2 \end{pmatrix}, \qquad (g^{\bar{\mu}\bar{\nu}}) \equiv \begin{pmatrix} g^{\mu\nu} & 0 \\ 0 & -\frac{1}{\phi^2} \end{pmatrix}. \tag{4.1}$$

In order to obtain non-vanishing wave-particle rest masses we have to consider a non-compactified theory, assuming that $g_{\mu\nu}$ and ϕ depend both on x^α and on x^4. The non-vanishing connection coefficients arising from the 5-*dimensional* metric tensor (4.1), given by:

$$\Gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\nu}} = \tfrac{1}{2} g^{\bar{\alpha}\bar{\beta}} \Big(g_{\bar{\mu}\bar{\beta},\bar{\nu}} + g_{\bar{\nu}\bar{\beta},\bar{\mu}} - g_{\bar{\mu}\bar{\nu},\bar{\beta}} \Big), \tag{4.2}$$

become:

$$\Gamma^{\alpha}_{\mu\nu} \equiv \tfrac{1}{2} g^{\alpha\beta}(g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta}) = \Gamma^{\alpha}_{<4>\mu\nu}, \tag{4.3}$$

$$\Gamma^{4}_{\mu\nu} \equiv \tfrac{1}{2} g^{44}(g_{\mu4,\nu} + g_{\nu4,\mu} - g_{\mu\nu,4}) = \tfrac{1}{2\phi^2}\, g_{\mu\nu,4}, \tag{4.4}$$

$$\Gamma^{\alpha}_{\mu4} \equiv \tfrac{1}{2} g^{\alpha\beta}(g_{\mu\beta,4} + g_{4\beta,\mu} - g_{\mu4,\beta}) = \tfrac{1}{2} g^{\alpha\beta} g_{\mu\beta,4}, \tag{4.5}$$

$$\Gamma^{\alpha}_{44} \equiv \tfrac{1}{2} g^{\alpha\beta}(g_{4\beta,4} + g_{4\beta,4} - g_{44,\beta}) = \phi\, g^{\alpha\beta} \phi_{,\beta}, \tag{4.6}$$

$$\Gamma^{4}_{\mu4} \equiv \tfrac{1}{2} g^{44}(g_{\mu4,4} + g_{44,\mu} - g_{\mu4,4}) = \tfrac{1}{\phi}\phi_{,\mu}, \tag{4.7}$$

$$\Gamma^{4}_{44} \equiv \tfrac{1}{2} g^{44}(g_{44,4} + g_{44,4} - g_{44,4}) = \tfrac{1}{\phi}\phi_{,4}, \tag{4.8}$$

where the label $_{<4>}$ denotes quantities related to the ordinary 4-*dimensional* space-time.

From the latter coefficients one is able to evaluate the Ricci tensor components:

$$R_{\bar{\mu}\bar{\nu}} = \Gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\nu},\bar{\alpha}} - \Gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\alpha},\bar{\nu}} - \Gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\beta}} \Gamma^{\bar{\beta}}_{\bar{\nu}\bar{\alpha}} + \Gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\nu}} \Gamma^{\bar{\beta}}_{\bar{\alpha}\bar{\beta}}. \tag{4.9}$$

In particular we are interested in R_{44}:

$$R_{44} = \Gamma^{\bar{\alpha}}_{44,\bar{\alpha}} - \Gamma^{\bar{\alpha}}_{4\bar{\alpha},4} - \Gamma^{\bar{\alpha}}_{4\bar{\beta}} \Gamma^{\bar{\beta}}_{4\bar{\alpha}} + \Gamma^{\bar{\alpha}}_{44} \Gamma^{\bar{\beta}}_{\bar{\alpha}\bar{\beta}}, \tag{4.10}$$

in order to arrive at the field equation for the scalar field ϕ. We have:

$$\begin{aligned} R_{44} = (\phi\, g^{\alpha\beta}\,\phi_{,\beta})_{,\alpha} - (\tfrac{1}{2}\,g^{\alpha\beta}\,g_{\alpha\beta,4})_{,4} - (\tfrac{1}{2}\,g^{\alpha\beta}\,g_{\gamma\beta,4})(\tfrac{1}{2}\,g^{\gamma\delta}\,g_{\alpha\delta,4}) - \\ -2(\tfrac{1}{\phi}\,\phi_{,\beta})(\phi\, g^{\beta\gamma}\,\phi_{,\gamma}) + (\phi\, g^{\alpha\beta}\,\phi_{,\beta})\mathit{\Gamma}^{\beta}_{<4>\alpha\beta} + \\ +\tfrac{1}{2\phi}\,\phi_{,4}\,g^{\alpha\beta}\,g_{\alpha\beta,4} + g^{\alpha\beta}\,\phi_{,\beta}\,\phi_{,\alpha}. \end{aligned} \quad (4.11)$$

After simplification it results:

$$\begin{aligned} R_{44} = \phi\, g^{\alpha\beta}\,\phi_{,\alpha|\beta} - \tfrac{1}{2}\,g^{\alpha\beta}\,g_{\alpha\beta,4,4} - \\ -\tfrac{1}{4}\,g^{\alpha\beta}_{\ \ ,4}\,g_{\alpha\beta,4} + \tfrac{1}{4\phi}\,\phi_{,4}\,g^{\alpha\beta}\,g_{\alpha\beta,4}, \end{aligned} \quad (4.12)$$

where $_|$ denotes here the covariant derivative respect to the connection $\boldsymbol{\Gamma}_{<4>}$ in V^4.

The equation for the scalar field ϕ:

$$R_{44} = 0, \quad (4.13)$$

becomes now explicitly (when the cosmological constant is assumed to be zero):

$$\phi\, g^{\alpha\beta}\,\phi_{,\alpha|\beta} - \tfrac{1}{2}\,g^{\alpha\beta}\,g_{\alpha\beta,4,4} - \tfrac{1}{4}\,g^{\alpha\beta}_{\ \ ,4}\,g_{\alpha\beta,4} + \tfrac{1}{4\phi}\,\phi_{,4}\,g^{\alpha\beta}\,g_{\alpha\beta,4} = 0. \quad (4.14)$$

We would have expected that a Klein-Gordon equation could arise from the last equation but we have obtained a different result. In fact:

1. Either one assumes to work with a compactified x^4 co-ordinate, then the dependence of all fields on x^4 disappears and (4.14) becomes a D'Alembertian equation:

$$g^{\alpha\beta}\,\phi_{,\alpha|\beta} = 0, \quad (4.15)$$

governing massless wave-particles in V^4;

2. Or no compactification is assumed to hold and then (4.14) can be reduced to a Klein-Gordon equation only imposing the strange condition:

$$\frac{1}{2\phi}g^{\alpha\beta}g_{\alpha\beta,4,4} + \frac{1}{4\phi}g^{\alpha\beta}_{\ \ ,4}g_{\alpha\beta,4} - \frac{1}{2\phi^2}g^{\alpha\beta}g_{\alpha\beta,4}\phi_{,4} = \frac{m^2c^2}{\hbar^2}\phi, \tag{4.16}$$

which results physically very improbable and difficult to be interpreted.

We observe that, when the cosmological constant is not zero a non-vanishing mass could be actually allowed.

In fact, from the trace of the Einstein equations in V^5 empty space-time:

$$R_{\bar{\mu}\bar{\nu}} - \frac{1}{2}Rg_{\bar{\mu}\bar{\nu}} - \Lambda g_{\bar{\mu}\bar{\nu}} = 0,$$

we can solve by contraction of the indices:

$$R = -\frac{10}{3}\Lambda, \tag{4.17}$$

so reducing the field equations to the simpler form:

$$R_{\bar{\mu}\bar{\nu}} + \frac{2}{3}\Lambda g_{\bar{\mu}\bar{\nu}} = 0. \tag{4.18}$$

Then the equation for the scalar field ϕ becomes:

$$R_{44} + \frac{2}{3}\Lambda g_{44} = 0, \tag{4.19}$$

which, when the co-ordintate x^4 is compactified, leads to the Klein-Gordon equation:

$$g^{\alpha\beta}\phi_{,\alpha|\beta} - \frac{2}{3}\Lambda\phi = 0, \tag{4.20}$$

where the particle mass is given by the identification:

$$\frac{mc^2}{\hbar^2} = \frac{2}{3}\Lambda, \tag{4.21}$$

from which we solve:

$$m = \frac{\hbar}{c}\sqrt{\frac{2}{3}\Lambda}. \tag{4.22}$$

Unphysically all the wave-particles would have the same mass!

Addition of the electromagnetic field according to the complete Kaluza-Klein theory would lead to even more complicated and unnatural results.

Therefore we are led to abandon this approach and to investigate an alternative way to field unification which will be the topic of the following chapters.

Chapter 5

An Alternative to Kaluza-Klein Theory

Abstract
In this chapter we present an original alternative approach to Kaluza-Klein theory. We consider each fundamental interaction field as governed by a vector potential which is an eigenvector of the metric tensor of some multidimensional space-time manifold V^n. We show that from Einstein equations governing the gravitational field in empty V^n space-time one is able to obtain, in V^4, the field equations for each fundamental interaction field, i.e., the Maxwellian equations both for Abelian and non-Abelian fields, and more the Einstein equations in presence of matter, with the expected energy-momentum tensor.

5.1 Introduction

Starting from the present chapter we present an alternative approach to the previously exploited usual Kaluza-Klein way to field unification. Our proposal is based on the idea of associating each fundamental interaction field to a vector potential which is en eigenvector of the metric tensor of some multidimensional space-time manifold V^n.

As we will see in an n-*dimensional* space-time the metric tensor, when represented onto the basis of its eigenvectors, leads to connection coefficients involving a 2-*index* antisymmetric tensor of the same form as a non-Abelian Maxwellian tensor. Abelian fields arise as a special case when the structure constants vanish. In the present chapter we examine such a formulation of *general relativity* in n dimensions, while the next chapter will be devoted to provide a physical interpretation of the theory so that all the known fundamental interactions (*i.e., gravitational, electro-weak* and *strong*) may be included within the metric and connection. As we will see a 16-*dimensional* space-time will be required in order to fit the *standard model* of elementary particles. (For a review on elementary particle standard model see, *e.g.,* [20], [24]).

5.2 Representations onto the Metric Eigenvectors

Let us consider an n-*dimensional* manifold V^n, endowed with a symmetric metric $\boldsymbol{g}$ of signature $(+, -, \cdots, -)$ and a torsionless connection $\boldsymbol{\Gamma}$. In a generic local frame $\mathcal{S}$ each point $\boldsymbol{x}$ is identified in V^n by its co-ordinates $x^{\bar{\mu}}$ with $\bar{\mu} = 0, \bar{i} = 1, 2, \cdots, n-1$. As usual x^0 is interpreted as the physically observable time co-ordinate, while the $x^{\bar{i}}$ are interpreted as space co-ordinates. According to the convention established in the previous chapters the non-underlined indices like $\mu = 0, i = 1, 2, 3,$ are reserved to the physically observable components, while the underlined Latin ones like, *e.g.*, $\underline{k} = 4, 5, \cdots, n-1$ characterize the non observable *extra* components. The invariant interval is defined in V^n by:

$$\mathrm{d}\bar{s}^2 = g_{\bar{\mu}\bar{\nu}}\,\mathrm{d}x^{\bar{\mu}}\,\mathrm{d}x^{\bar{\nu}}, \tag{5.1}$$

where $g_{\bar{\mu}\bar{\nu}}$ are the components of the metric tensor relative to the frame $\mathcal{S}$. As usual, summation under repeated indices is intended according to Einstein convention. The components of the connection are given by:

$$\Gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\nu}} = \tfrac{1}{2} g^{\bar{\alpha}\bar{\beta}} \left(g_{\bar{\mu}\bar{\beta},\bar{\nu}} + g_{\bar{\nu}\bar{\beta},\bar{\mu}} - g_{\bar{\mu}\bar{\nu},\bar{\beta}} \right). \tag{5.2}$$

The components of the inverse metric tensor $g^{\bar{\alpha}\bar{\beta}}$ are defined, as usual, by the relation:

$$g^{\bar{\alpha}\bar{\beta}}\, g_{\bar{\beta}\bar{\gamma}} = \delta^{\bar{\alpha}}_{\bar{\gamma}}\,. \tag{5.3}$$

5.2.1 Representations of the Metric and Connection

The metric tensor, being symmetric, admits n real eigenvalues and n real linearly independent eigenvectors (*basis*) $\{a^{\bar{\mu}}_{(\bar{\sigma})},\ \bar{\sigma} = 0, 1, 2, \cdots, n-1\}$ fulfilling the orthonormality conditions:

$$g_{\bar{\mu}\bar{\nu}}\, a^{\bar{\mu}}_{(\bar{\sigma})}\, a^{\bar{\nu}}_{(\bar{\tau})} = \eta_{(\bar{\sigma})(\bar{\tau})}, \qquad \bar{\sigma}, \bar{\tau} = 0, 1, 2, \cdots, n-1, \tag{5.4}$$

where, here, $(\eta_{(\bar{\sigma})(\bar{\tau})}) \equiv \mathit{diag}(1, -1, \cdots, -1)$.

It follows that:

$$g_{\bar{\mu}\bar{\nu}} = \eta_{(\bar{\sigma})(\bar{\tau})}\, a^{(\bar{\sigma})}_{\bar{\mu}}\, a^{(\bar{\tau})}_{\bar{\nu}} \equiv a_{(\bar{\sigma})\bar{\mu}}\, a^{(\bar{\sigma})}_{\bar{\nu}}, \tag{5.5}$$

is the representation of the metric tensor on the basis of its eigenvectors. We observe that these eigenvectors are not univocally defined by the orthonormality condition (5.4) since they are undetermined by an imaginary exponential factor $e^{i\theta}$ which leaves unchanged the *real* metric tensor components, provided that the complex scalar product $a^*_{(\bar{\sigma})\bar{\mu}}a^{(\bar{\sigma})}_{\bar{\nu}}$ replaces the real product $a_{(\bar{\sigma})\bar{\mu}}a^{(\bar{\sigma})}_{\bar{\nu}}$.

As we will see in chapters 8 and 9 the latter degree of freedom will be of main relevance in order to obtain periodic wave propagating solutions for the fields $a^{(\bar{\sigma})}_{\bar{\mu}}$.

In the following we will drop (leaving it as understood) the * complex conjugation operator, when it is not necessarily required, in order to avoid too heavy notations.

The representation of the connection coefficients relative to the basis $\{a^{(\bar{\sigma})}_{\bar{\mu}}\}$ follows from the previous definitions:

$$\begin{aligned}\Gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\nu}} = \tfrac{1}{2}g^{\bar{\alpha}\bar{\beta}}\Big[&\Big(a^{(\bar{\sigma})}_{\bar{\mu},\bar{\nu}} + a^{(\bar{\sigma})}_{\bar{\nu},\bar{\mu}}\Big)a_{(\bar{\sigma})\bar{\beta}} + \\ &+a_{(\bar{\sigma})\bar{\mu}}\Big(a^{(\bar{\sigma})}_{\bar{\beta},\bar{\nu}} - a^{(\bar{\sigma})}_{\bar{\nu},\bar{\beta}}\Big) + \Big(a^{(\bar{\sigma})}_{\bar{\beta},\bar{\mu}} - a^{(\bar{\sigma})}_{\bar{\mu},\bar{\beta}}\Big)a_{(\bar{\sigma})\bar{\nu}}\Big].\end{aligned} \tag{5.6}$$

It is remarkable that possible non-Abelian contributions may be added and subtracted without altering the previous result:

$$\begin{aligned}\Gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\nu}} = \tfrac{1}{2}g^{\bar{\alpha}\bar{\beta}}\Big[&\Big(a^{(\bar{\sigma})}_{\bar{\mu},\bar{\nu}} + a^{(\bar{\sigma})}_{\bar{\nu},\bar{\mu}}\Big)a_{(\bar{\sigma})\bar{\beta}} + \\ &+a_{(\bar{\sigma})\bar{\mu}}\Big(a^{(\bar{\sigma})}_{\bar{\beta},\bar{\nu}} - a^{(\bar{\sigma})}_{\bar{\nu},\bar{\beta}} + C^{(\bar{\sigma})}_{\ (\bar{\tau})(\bar{\upsilon})}a^{(\bar{\tau})}_{(\bar{\beta})}a^{(\bar{\upsilon})}_{\bar{\nu}}\Big) + \\ &+\Big(a^{(\bar{\sigma})}_{\bar{\beta},\bar{\mu}} - a^{(\bar{\sigma})}_{\bar{\mu},\bar{\beta}} + C^{(\bar{\sigma})}_{\ (\bar{\tau})(\bar{\upsilon})}a^{(\bar{\tau})}_{(\bar{\beta})}a^{(\bar{\upsilon})}_{\bar{\mu}}\Big)a_{(\bar{\sigma})\bar{\nu}}\Big],\end{aligned} \tag{5.7}$$

since:

$$C^{(\bar{\sigma})}_{\ (\bar{\tau})(\bar{\upsilon})}a_{(\bar{\sigma})\bar{\mu}}a^{(\bar{\tau})}_{\bar{\beta}}a^{(\bar{\upsilon})}_{\bar{\nu}} + C^{(\bar{\sigma})}_{\ (\bar{\tau})(\bar{\upsilon})}a_{(\bar{\sigma})\bar{\nu}}a^{(\bar{\tau})}_{\bar{\beta}}a^{(\bar{\upsilon})}_{\bar{\mu}} = 0, \tag{5.8}$$

because of total antisymmetry of the *structure constants* $C^{(\bar{\sigma})}_{\ (\bar{\tau})(\bar{\upsilon})}$ characterizing non-Abelian field theories. (On non-Abelian field theories, beside the original paper by Yang and Mills [32], see *e.g.,* [10]).

Introducing the antisymmetric tensors:

$$f^{(\bar{\sigma})}_{\bar{\mu}\bar{\nu}} = a^{(\bar{\sigma})}_{\bar{\nu},\bar{\mu}} - a^{(\bar{\sigma})}_{\bar{\mu},\bar{\nu}} + C^{(\bar{\sigma})}_{(\bar{\tau})(\bar{\upsilon})} a^{(\bar{\tau})}_{\bar{\mu}} a^{(\bar{\upsilon})}_{\bar{\nu}}, \tag{5.9}$$

and the symbols (which manifestly are not Riemannian tensors):

$$h^{(\bar{\sigma})}_{\bar{\mu}\bar{\nu}} = a^{(\bar{\sigma})}_{\bar{\mu},\bar{\nu}} + a^{(\bar{\sigma})}_{\bar{\nu},\bar{\mu}}, \tag{5.10}$$

and taking into account the orthonormalization relation (5.4), we reach the following representation of the connection coefficients:

$$\Gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\nu}} = \frac{1}{2}\Big(a^{\bar{\alpha}}_{(\bar{\sigma})} h^{(\bar{\sigma})}_{\bar{\mu}\bar{\nu}} - a^{(\bar{\sigma})}_{\bar{\mu}} f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\nu}}} - f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\mu}}} a^{(\bar{\sigma})}_{\bar{\nu}}\Big), \tag{5.11}$$

where:

$$f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\mu}}} = g^{\bar{\alpha}\bar{\beta}} f_{(\bar{\sigma})\bar{\beta}\bar{\mu}}. \tag{5.12}$$

It is relevant that the connection coefficients involve antisymmetric tensors the structure of which is the same as that of non-Abelian Yang-Mills fields together with a symmetric non-tensor additional field $h^{(\bar{\sigma})}_{\bar{\mu}\bar{\nu}}$.

Let us now introduce a new symbol which we could name *"reduced connection"* defined by:

$$\gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\nu}} = \frac{1}{2} a^{\bar{\alpha}}_{(\bar{\sigma})} h^{(\bar{\sigma})}_{\bar{\mu}\bar{\nu}}. \tag{5.13}$$

Finally we are led to represent the *full* connection coefficient in V^n as:

$$\Gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\nu}} = \gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\nu}} - \frac{1}{2}\Big(a^{(\bar{\sigma})}_{\bar{\mu}} f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\nu}}} + f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\mu}}} a^{(\bar{\sigma})}_{\bar{\nu}}\Big). \tag{5.14}$$

The latter representation of $\Gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\nu}}$ means that the gravitational strength in V^n is generated by a tensor contribution depending on the non-Abelian fields $f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\mu}}}$, the vector potential $a^{(\bar{\sigma})}_{\bar{\mu}}$, and the reduced connection $\gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\nu}}$.

The presence of the antisymmetric tensors $f^{\bar\alpha}_{(\bar\sigma)\,\bar\mu}$, in the connection coefficients reasonably suggests that, when the space-time dimensionality is greater than 4, the *electro-weak* and *strong interaction fields* may be included into the metric tensor in a unified field theory. We will follow this suggestion.

5.2.2 Representation of the Ricci Tensor

The Ricci tensor is given by:

$$R_{\bar\mu\bar\nu} = \Gamma^{\bar\alpha}_{\bar\mu\bar\nu,\bar\alpha} - \Gamma^{\bar\alpha}_{\bar\mu\bar\alpha,\bar\nu} - \Gamma^{\bar\alpha}_{\bar\mu\bar\beta}\,\Gamma^{\bar\beta}_{\bar\nu\bar\alpha} + \Gamma^{\bar\alpha}_{\bar\mu\bar\nu}\,\Gamma^{\bar\beta}_{\bar\alpha\bar\beta}. \tag{5.15}$$

In order to be able to manage calculations it proves convenient to introduce the following notations:

$$\Gamma^{\bar\alpha}_{\bar\mu\bar\nu} = \gamma^{\bar\alpha}_{\bar\mu\bar\nu} + \mathcal{G}^{\bar\alpha}_{\bar\mu\bar\nu}, \tag{5.16}$$

$$\mathcal{G}^{\bar\alpha}_{\bar\mu\bar\nu} = -\tfrac{1}{2}\Big(a^{(\bar\sigma)}_{\bar\mu} f^{\bar\alpha}_{(\bar\sigma)\,\bar\nu} + f^{\bar\alpha}_{(\bar\sigma)\,\bar\mu}\, a^{(\bar\sigma)}_{\bar\nu}\Big), \tag{5.17}$$

from which we have also:

$$\Gamma^{\bar\alpha}_{\bar\mu\bar\alpha} = \gamma^{\bar\alpha}_{\bar\mu\bar\alpha} + \mathcal{G}^{\bar\alpha}_{\bar\mu\bar\alpha},\;\; \mathcal{G}^{\bar\alpha}_{\bar\mu\bar\alpha} = -\tfrac{1}{2} f^{\bar\alpha}_{(\bar\sigma)\,\bar\mu}\, a^{(\bar\sigma)}_{\bar\alpha},\;\; g^{\bar\mu\bar\nu}\,\mathcal{G}^{\bar\alpha}_{\bar\mu\bar\nu} = -a^{(\bar\sigma)}_{\bar\mu}\, f^{\bar\alpha\bar\mu}_{(\bar\sigma)}, \tag{5.18}$$

thanks to the symmetries. The Ricci tensor can now be written in the form:

$$R_{\bar\mu\bar\nu} = \widetilde{R}_{\bar\mu\bar\nu} + \mathcal{G}^{\bar\alpha}_{\bar\mu\bar\nu:\bar\alpha} - \mathcal{G}^{\bar\alpha}_{\bar\mu\bar\alpha:\bar\nu} - \mathcal{G}^{\bar\alpha}_{\bar\mu\bar\beta}\,\mathcal{G}^{\bar\beta}_{\bar\nu\bar\alpha} + \mathcal{G}^{\bar\alpha}_{\bar\mu\bar\nu}\,\mathcal{G}^{\bar\beta}_{\bar\alpha\bar\beta}, \tag{5.19}$$

the trace of which results to be:

$$R = \widetilde{R} + g^{\bar\mu\bar\nu}\Big(\mathcal{G}^{\bar\alpha}_{\bar\mu\bar\nu:\bar\alpha} - \mathcal{G}^{\bar\alpha}_{\bar\mu\bar\alpha:\bar\nu} - \mathcal{G}^{\bar\alpha}_{\bar\mu\bar\beta}\,\mathcal{G}^{\bar\beta}_{\bar\nu\bar\alpha} + \mathcal{G}^{\bar\alpha}_{\bar\mu\bar\nu}\,\mathcal{G}^{\bar\beta}_{\bar\alpha\bar\beta}\Big). \tag{5.20}$$

The tensor:

$$\widetilde{R}_{\bar\mu\bar\nu} = \gamma^{\bar\alpha}_{\bar\mu\bar\nu,\bar\alpha} - \gamma^{\bar\alpha}_{\bar\mu\bar\alpha,\bar\nu} - \gamma^{\bar\alpha}_{\bar\mu\bar\beta}\,\gamma^{\bar\beta}_{\bar\nu\bar\alpha} + \gamma^{\bar\alpha}_{\bar\mu\bar\nu}\,\gamma^{\bar\beta}_{\bar\alpha\bar\beta}, \tag{5.21}$$

we have just introduced, will be named the *"reduced Ricci tensor"*, being evaluated respect to the *reduced connection*. So we have:

$$\mathcal{G}^{\bar\alpha}_{\bar\mu\bar\nu:\bar\alpha} = \mathcal{G}^{\bar\alpha}_{\bar\mu\bar\nu,\bar\alpha} - \gamma^{\bar\beta}_{\bar\mu\bar\alpha}\mathcal{G}^{\bar\alpha}_{\bar\nu\bar\beta} - \gamma^{\bar\beta}_{\bar\nu\bar\alpha}\mathcal{G}^{\bar\alpha}_{\bar\mu\bar\beta} + \gamma^{\bar\alpha}_{\bar\beta\bar\alpha}\mathcal{G}^{\bar\beta}_{\bar\mu\bar\nu}, \tag{5.22}$$

and:

$$\mathcal{G}^{\bar\alpha}_{\bar\mu\bar\alpha:\bar\nu} = \mathcal{G}^{\bar\alpha}_{\bar\mu\bar\alpha,\bar\nu} - \gamma^{\bar\beta}_{\bar\mu\bar\nu}\mathcal{G}^{\bar\alpha}_{\bar\beta\bar\alpha}. \tag{5.23}$$

We have denoted by a colon (:) the covariant derivative evaluated respect to the *reduced connection,* which will be named the *"reduced covariant derivative",* to be distinguished by the usual covariant derivative denoted by semicolon (;), which is evaluated respect to the *full* connection. Replacing the *full* connection $\boldsymbol{\Gamma} \equiv (\Gamma^{\bar\alpha}_{\bar\mu\bar\nu})$ by the reduced connection $\gamma \equiv (\gamma^{\bar\alpha}_{\bar\mu\bar\nu})$ in evaluating covariant derivatives means that the gravitational field in V^n is only partially treated as hidden within the geometry of space-time, its remaining part being still described by physical fields the stress tensor of which are $\boldsymbol{f}^{(\bar\sigma)} \equiv (f^{(\bar\sigma)}_{\bar\mu\bar\nu})$.

Now, in order to evaluate the Ricci tensor we proceed in two steps: (a) we calculate $\mathcal{G}^{\bar\alpha}_{\bar\mu\bar\nu:\bar\alpha}$ and (b) we evaluate the quadratic contributions $G^{\bar\alpha}_{\bar\mu\bar\beta}\mathcal{G}^{\bar\beta}_{\bar\nu\bar\alpha}$ and $\mathcal{G}^{\bar\alpha}_{\bar\mu\bar\nu}\mathcal{G}^{\bar\beta}_{\bar\alpha\bar\beta}$.

1. Developing calculations in (5.22) the divergence term becomes:

$$\begin{aligned}\mathcal{G}^{\bar\alpha}_{\bar\mu\bar\nu:\bar\alpha} &= -\tfrac{1}{2}\Big(a^{(\bar\sigma)}_{\bar\mu} f^{\bar\alpha}_{(\bar\sigma)_{\bar\nu:\bar\alpha}} + f^{\bar\alpha}_{(\bar\sigma)_{\bar\mu:\bar\alpha}} a^{(\bar\sigma)}_{\bar\nu}\Big) - \\ &-\tfrac{1}{2}\Big(a^{(\bar\sigma)}_{\bar\mu:\bar\alpha} f^{\bar\alpha}_{(\bar\sigma)_{\bar\nu}} + f^{\bar\alpha}_{(\bar\sigma)_{\bar\mu}} a^{(\bar\sigma)}_{\bar\nu:\bar\alpha}\Big).\end{aligned} \tag{5.24}$$

We observe that since:

$$a^{(\bar\sigma)}_{\bar\mu:\bar\alpha} = f^{(\bar\sigma)}_{\bar\alpha\bar\mu} + a^{(\bar\sigma)}_{\bar\alpha:\bar\mu}, \tag{5.25}$$

in (5.24) we can write equivalently:

$$\mathcal{G}^{\bar{\alpha}}_{\bar{\mu}\bar{\nu}:\bar{\alpha}} = -\tfrac{1}{2}\left(a^{(\bar{\sigma})}_{\bar{\mu}} f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\nu}:\bar{\alpha}}} + f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\mu}:\bar{\alpha}}} a^{(\bar{\sigma})}_{\bar{\nu}}\right) - \\ -f^{(\bar{\sigma})}_{\bar{\alpha}\bar{\mu}} f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\nu}}} - \tfrac{1}{2}\left(a^{(\bar{\sigma})}_{\bar{\alpha}:\bar{\mu}} f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\nu}}} + f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\mu}}} a^{(\bar{\sigma})}_{\bar{\alpha}:\bar{\nu}}\right). \tag{5.26}$$

2. The quadratic contributions assume the following explicit form:

$$G^{\bar{\alpha}}_{\bar{\mu}\bar{\beta}}\mathcal{G}^{\bar{\beta}}_{\bar{\nu}\bar{\alpha}} = \tfrac{1}{4}a^{(\bar{\sigma})}_{\bar{\mu}} f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\beta}}} f^{\bar{\beta}}_{(\bar{\tau})_{\bar{\alpha}}} a^{(\bar{\tau})}_{\bar{\nu}} + \tfrac{1}{4}a^{(\bar{\sigma})}_{\bar{\mu}} f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\beta}}} f^{\bar{\beta}}_{(\bar{\tau})_{\bar{\nu}}} a^{(\bar{\tau})}_{\bar{\alpha}} + \\ +\tfrac{1}{4}a^{(\bar{\sigma})}_{\bar{\beta}} f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\mu}}} f^{\bar{\beta}}_{(\bar{\tau})_{\bar{\alpha}}} a^{(\bar{\tau})}_{\bar{\nu}} + \tfrac{1}{4} f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\mu}}} a^{(\bar{\sigma})}_{\bar{\beta}} f^{\bar{\beta}}_{(\bar{\tau})_{\bar{\nu}}} a^{(\bar{\tau})}_{\bar{\alpha}}, \tag{5.27}$$

$$\mathcal{G}^{\bar{\alpha}}_{\bar{\mu}\bar{\nu}}\mathcal{G}^{\bar{\beta}}_{\bar{\alpha}\bar{\beta}} = \tfrac{1}{4}\left(a^{(\bar{\sigma})}_{\bar{\mu}} f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\nu}}} + f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\mu}}} a^{(\bar{\sigma})}_{\bar{\nu}}\right) f^{\bar{\beta}}_{(\bar{\tau})_{\bar{\alpha}}} a^{(\bar{\tau})}_{\bar{\beta}}. \tag{5.28}$$

It is remarkable that when it happens that:

$$f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\mu}}} a^{(\bar{\sigma})}_{\bar{\alpha}} = 0, \tag{5.29}$$

we gain great simplification of the previous results.

In fact it results:

$$\Gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\alpha}} = \gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\alpha}}, \tag{5.30}$$

$$\mathcal{G}^{\bar{\alpha}}_{\bar{\mu}\bar{\alpha}} = 0, \tag{5.31}$$

$$g^{\bar{\mu}\bar{\nu}}\mathcal{G}^{\bar{\alpha}}_{\bar{\mu}\bar{\nu}} = 0. \tag{5.32}$$

Therefore the Ricci tensor assumes the simplified form:

$$R_{\bar{\mu}\bar{\nu}} = \widetilde{R}_{\bar{\mu}\bar{\nu}} + \mathcal{G}^{\bar{\alpha}}_{\bar{\mu}\bar{\nu}:\bar{\alpha}} - \mathcal{G}^{\bar{\alpha}}_{\bar{\mu}\bar{\beta}}\mathcal{G}^{\bar{\beta}}_{\bar{\nu}\bar{\alpha}}, \tag{5.33}$$

which can be written more explicitly as:

$$R_{\bar{\mu}\bar{\nu}} = \widetilde{R}_{\bar{\mu}\bar{\nu}} - \tfrac{1}{2}\left(a^{(\bar{\sigma})}_{\bar{\mu}} f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\nu}:\bar{\alpha}}} + f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\mu}:\bar{\alpha}}} a^{(\bar{\sigma})}_{\bar{\nu}}\right) - \\ -f^{(\bar{\sigma})}_{\bar{\alpha}\bar{\mu}} f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\nu}}} - \tfrac{1}{2}\left(a^{(\bar{\sigma})}_{\bar{\alpha}:\bar{\mu}} f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\nu}}} + f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\mu}}} a^{(\bar{\sigma})}_{\bar{\alpha}:\bar{\nu}}\right) - \mathcal{G}^{\bar{\alpha}}_{\bar{\mu}\bar{\beta}}\mathcal{G}^{\bar{\beta}}_{\bar{\nu}\bar{\alpha}}. \tag{5.34}$$

If we introduce the notation:

$$\mathcal{J}_{\bar{\mu}\bar{\nu}} = -\tfrac{1}{2}\Big(a^{(\bar{\sigma})}_{\bar{\alpha}:\bar{\mu}}\, f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\nu}}} + f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\mu}}}\, a^{(\bar{\sigma})}_{\bar{\alpha}:\bar{\nu}}\Big) - \\ -\tfrac{1}{2(n-2)}\, f^{(\bar{\sigma})}_{\bar{\alpha}\bar{\beta}}\, f^{\bar{\alpha}\bar{\beta}}_{(\bar{\sigma})}\, g_{\bar{\mu}\bar{\nu}} - \mathcal{G}^{\bar{\alpha}}_{\bar{\mu}\bar{\beta}}\, \mathcal{G}^{\bar{\beta}}_{\bar{\nu}\bar{\alpha}}, \tag{5.35}$$

we can write equivalently:

$$R_{\bar{\mu}\bar{\nu}} = \widetilde{R}_{\bar{\mu}\bar{\nu}} - \tfrac{1}{2}\Big(a^{(\bar{\sigma})}_{\bar{\mu}}\, f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\nu}:\bar{\alpha}}} + f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\mu}:\bar{\alpha}}}\, a^{(\bar{\sigma})}_{\bar{\nu}}\Big) - \\ -f^{(\bar{\sigma})}_{\bar{\alpha}\bar{\mu}}\, f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\nu}}} + \tfrac{1}{2(n-2)}\, f^{(\bar{\sigma})}_{\bar{\alpha}\bar{\beta}}\, f^{\bar{\alpha}\bar{\beta}}_{(\bar{\sigma})}\, g_{\bar{\mu}\bar{\nu}} + \mathcal{J}_{\bar{\mu}\bar{\nu}}\,. \tag{5.36}$$

We point out that the tensor $\mathcal{J}_{\bar{\mu}\bar{\nu}}$ may be suitably represented as:

$$\mathcal{J}_{\bar{\mu}\bar{\nu}} = \tfrac{1}{2}\Big[a^{(\bar{\sigma})}_{\bar{\mu}}\Big(\mathcal{J}_{(\bar{\sigma})\bar{\nu}} + \mathcal{A}_{(\bar{\sigma})\bar{\nu}}\Big) + a^{(\bar{\sigma})}_{\bar{\nu}}\Big(\mathcal{J}_{(\bar{\sigma})\bar{\mu}} + \mathcal{A}_{(\bar{\sigma})\bar{\mu}}\Big)\Big], \tag{5.37}$$

with:

$$\mathcal{J}_{(\bar{\sigma})\bar{\mu}} = \mathcal{J}_{\bar{\mu}\bar{\nu}}\, a^{\bar{\nu}}_{(\bar{\sigma})}, \qquad \mathcal{A}_{(\bar{\sigma})\bar{\mu}} = \mathcal{A}_{\bar{\mu}\bar{\nu}}\, a^{\bar{\nu}}_{(\bar{\sigma})}, \tag{5.38}$$

where $\mathcal{A}_{\bar{\mu}\bar{\nu}}$ is an arbitrary antisymmetric tensor. In fact the additional term adds an identically vanishing contribution to $\mathcal{J}_{\bar{\mu}\bar{\nu}}$ (and then to the Ricci tensor) since:

$$\mathcal{A}_{\bar{\nu}\bar{\rho}}\, a^{\bar{\rho}}_{(\bar{\sigma})}\, a^{(\bar{\sigma})}_{\bar{\mu}} + \mathcal{A}_{\bar{\mu}\bar{\rho}}\, a^{\bar{\rho}}_{(\bar{\sigma})} a^{(\bar{\sigma})}_{\bar{\nu}} \equiv \mathcal{A}_{\bar{\mu}\bar{\nu}} + \mathcal{A}_{\bar{\nu}\bar{\mu}} = 0, \tag{5.39}$$

thanks to antisymmetry, being:

$$a^{\bar{\rho}}_{(\bar{\sigma})}\, a^{(\bar{\sigma})}_{\bar{\mu}} = \delta^{\bar{\rho}}_{\bar{\mu}}\,. \tag{5.40}$$

As we will see in chapter 7 the tensor $\mathcal{A}_{\bar{\mu}\bar{\rho}}$ introduces a degree of freedom that will play an important role to fit elementary particle *standard model.*

Moreover we observe that we can replace the Riemannian covariant derivatives with non-Abelian covariant derivatives:

$$\mathrm{D}_{\bar{\alpha}}\, f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\mu}}} = f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\mu}:\bar{\alpha}}} + C^{(\bar{\upsilon})}_{\ (\bar{\sigma})(\bar{\tau})}\, a^{(\bar{\tau})}_{\bar{\alpha}}\, f^{\bar{\alpha}}_{(\bar{\upsilon})\bar{\mu}}, \tag{5.41}$$

i.e. we may write:

$$a_{\bar{\mu}}^{(\bar{\sigma})} f_{(\bar{\sigma})_{\bar{\nu}:\bar{\alpha}}}^{\bar{\alpha}} + f_{(\bar{\sigma})_{\bar{\mu}:\bar{\alpha}}}^{\bar{\alpha}} a_{\bar{\nu}}^{(\bar{\sigma})} \equiv a_{\bar{\mu}}^{(\bar{\sigma})} \mathrm{D}_{\bar{\alpha}} f_{(\bar{\sigma})_{\bar{\nu}}}^{\bar{\alpha}} + \mathrm{D}_{\bar{\alpha}} f_{(\bar{\sigma})_{\bar{\mu}}}^{\bar{\alpha}} a_{\bar{\nu}}^{(\bar{\sigma})}, \tag{5.42}$$

since:

$$a_{\bar{\mu}}^{(\bar{\sigma})} C^{(\bar{\upsilon})}{}_{(\bar{\sigma})(\bar{\tau})} a_{\bar{\alpha}}^{(\bar{\tau})} f_{(\bar{\upsilon})\bar{\mu}}^{\bar{\alpha}} + C^{(\bar{\upsilon})}{}_{(\bar{\sigma})(\bar{\tau})} a_{\bar{\alpha}}^{(\bar{\tau})} f_{(\bar{\upsilon})\bar{\nu}}^{\bar{\alpha}} a_{\bar{\mu}}^{(\bar{\sigma})} = 0. \tag{5.43}$$

Finally we obtain the meaningful representation:

$$\begin{aligned} R_{\bar{\mu}\bar{\nu}} = \widetilde{R}_{\bar{\mu}\bar{\nu}} - f_{\bar{\alpha}\bar{\mu}}^{(\bar{\sigma})} f_{(\bar{\sigma})_{\bar{\nu}}}^{\bar{\alpha}} + \tfrac{1}{2(n-2)} f_{\bar{\alpha}\bar{\beta}}^{(\bar{\sigma})} f_{(\bar{\sigma})}^{\bar{\alpha}\bar{\beta}} g_{\bar{\mu}\bar{\nu}} - \\ -\tfrac{1}{2} a_{\bar{\mu}}^{(\bar{\sigma})} \Big(\mathrm{D}_{\bar{\alpha}} f_{(\bar{\sigma})_{\bar{\nu}}}^{\bar{\alpha}} - \mathcal{J}_{(\bar{\sigma})\bar{\nu}} - \mathcal{A}_{(\bar{\sigma})\bar{\nu}} \Big) - \\ -\tfrac{1}{2} \Big(\mathrm{D}_{\bar{\alpha}} f_{(\bar{\sigma})_{\bar{\mu}}}^{\bar{\alpha}} - \mathcal{J}_{(\bar{\sigma})\bar{\mu}} - \mathcal{A}_{(\bar{\sigma})\bar{\mu}} \Big) a_{\bar{\nu}}^{(\bar{\sigma})}. \end{aligned} \tag{5.44}$$

Then the Ricci scalar is given by:

$$R = \widetilde{R} - \tfrac{n-4}{2(n-2)} f_{\bar{\alpha}\bar{\beta}}^{(\bar{\sigma})} f_{(\bar{\sigma})}^{\bar{\alpha}\bar{\beta}} - a^{(\bar{\sigma})\bar{\beta}} \Big(\mathrm{D}_{\bar{\alpha}} f_{(\bar{\sigma})_{\bar{\beta}}}^{\bar{\alpha}} - \mathcal{J}_{(\bar{\sigma})\bar{\beta}} - \mathcal{A}_{(\bar{\sigma})\bar{\beta}} \Big). \tag{5.45}$$

5.3 Field Equations

The Einstein field equations for the unknown components of $g_{\bar{\mu}\bar{\nu}}$, or its eigenvectors $a_{\bar{\mu}}^{(\bar{\sigma})}$ (*vector potentials*) become in general, in presence of a *cosmological constant*:

$$R_{\bar{\mu}\bar{\nu}} - \tfrac{1}{2} R g_{\bar{\mu}\bar{\nu}} - \Lambda g_{\bar{\mu}\bar{\nu}} = 0. \tag{5.46}$$

These field equations, written in a more expanded form, thanks to (5.19) and (5.20) appear as follows:

$$\begin{aligned} \widetilde{R}_{\bar{\mu}\bar{\nu}} - \tfrac{1}{2} \widetilde{R} g_{\bar{\mu}\bar{\nu}} + \mathcal{G}_{\bar{\mu}\bar{\nu}:\bar{\alpha}}^{\bar{\alpha}} - \mathcal{G}_{\bar{\mu}\bar{\alpha}:\bar{\nu}}^{\bar{\alpha}} - \mathcal{G}_{\bar{\mu}\bar{\beta}}^{\bar{\alpha}} \mathcal{G}_{\bar{\nu}\bar{\alpha}}^{\bar{\beta}} + \mathcal{G}_{\bar{\mu}\bar{\nu}}^{\bar{\alpha}} \mathcal{G}_{\bar{\alpha}\bar{\beta}}^{\bar{\beta}} - \\ -\tfrac{1}{2} g^{\gamma\delta} \Big(\mathcal{G}_{\gamma\delta:\bar{\alpha}}^{\bar{\alpha}} - \mathcal{G}_{\gamma\bar{\alpha}:\delta}^{\bar{\alpha}} - \mathcal{G}_{\gamma\bar{\beta}}^{\bar{\alpha}} \mathcal{G}_{\delta\bar{\alpha}}^{\bar{\beta}} + \mathcal{G}_{\gamma\delta}^{\bar{\alpha}} \mathcal{G}_{\bar{\alpha}\bar{\beta}}^{\bar{\beta}} \Big) g_{\bar{\mu}\bar{\nu}} - \Lambda g_{\bar{\mu}\bar{\nu}} = 0. \end{aligned} \tag{5.47}$$

Before we proceed in examining the field equations we need choose suitable additional conditions in order to determine the system of the Einstein equations. In the next section we will show that the previously found condition (5.29), which greatly simplifies the Ricci tensor is just equivalent to the well known Lorentz gauge for each vector potential $a_{\bar{\mu}}^{(\bar{\sigma})}$.

5.3.1 The Lorentz Gauge

The full expansion of the terms appearing in the equations (5.47) is very heavy but we can lighten it with a suitable gauge choice. In fact n conditions (as many as the space-time dimensionality) are needed to solve Einstein equations.

Here we can just impose the n conditions (5.29), which are scalar equations, being independent of the co-ordinate choice.

Now we observe that, because of the orthonormality of the metric tensor eigenvectors, we have:

$$g^{\bar{\alpha}\bar{\beta}}\,a_{(\bar{\sigma})\bar{\alpha}}\,a_{\bar{\beta}}^{(\bar{\sigma})} \equiv g^{\bar{\alpha}\bar{\beta}}\,g_{\bar{\alpha}\bar{\beta}} = n, \tag{5.48}$$

and thanks to the the metricity condition:

$$g^{\bar{\alpha}\bar{\beta};\bar{\mu}} \equiv g^{\bar{\alpha}\bar{\beta},\bar{\mu}} + \Gamma_{\rho\bar{\mu}}^{\bar{\alpha}}\,g^{\rho\bar{\beta}} + \Gamma_{\rho\bar{\mu}}^{\bar{\beta}}\,g^{\bar{\alpha}\rho} = 0, \tag{5.49}$$

remembering that semicolon denotes the covariant derivative respect to the *full* connection $\boldsymbol{\Gamma}$.

It follows:

$$\left(g^{\bar{\alpha}\bar{\beta}}\,a_{(\bar{\sigma})\bar{\alpha}}\,a_{\bar{\beta}}^{(\bar{\sigma})}\right)_{;\bar{\mu}} \equiv 2g^{\bar{\alpha}\bar{\beta}}\,a_{(\bar{\sigma})\bar{\alpha};\bar{\mu}}\,a_{\bar{\beta}}^{(\bar{\sigma})} \equiv 2a_{(\bar{\sigma})}^{\bar{\alpha}}\,a_{\bar{\alpha};\bar{\mu}}^{(\bar{\sigma})} = 0. \tag{5.50}$$

Thanks to orthonormalization condition it results also:

$$a_{(\bar{\sigma})}^{\bar{\alpha}}\,a_{\bar{\mu};\bar{\alpha}}^{(\bar{\sigma})} = -a_{(\bar{\sigma});\bar{\alpha}}^{\bar{\alpha}}\,a_{\bar{\mu}}^{(\bar{\sigma})}. \tag{5.51}$$

Thanks to the last results (5.51) we arrive at:

$$f^{\bar\alpha}_{(\bar\sigma)\,\bar\mu}\,a^{(\bar\sigma)}_{\bar\alpha} \equiv f^{(\bar\sigma)}_{\bar\alpha\bar\mu}\,a^{\bar\alpha}_{(\bar\sigma)} \equiv a^{\bar\alpha}_{(\bar\sigma)}\,a^{(\bar\sigma)}_{\bar\mu;\bar\alpha} - a^{\bar\alpha}_{(\bar\sigma)}\,a^{(\bar\sigma)}_{\bar\alpha;\bar\mu} = -a^{\bar\alpha}_{(\bar\sigma);\bar\alpha}\,a^{(\bar\sigma)}_{\bar\mu}. \tag{5.52}$$

And taking into account (5.50) it remains:

$$a^{\bar\alpha}_{(\bar\sigma)\,;\bar\alpha} = 0. \tag{5.53}$$

Therefore we may conclude that the conditions (5.29) are equivalent to the Lorentz gauge (5.53).

Now being $\mathcal{G}^{\bar\beta}_{\bar\alpha\bar\beta} = 0$, we have also:

$$a^{\bar\alpha}_{(\bar\sigma)\,;\bar\alpha} \equiv a^{\bar\alpha}_{(\bar\sigma)\,,\bar\alpha} + \Gamma^{\bar\alpha}_{\bar\alpha\bar\beta}\,a^{\bar\beta}_{(\bar\sigma)} \equiv a^{\bar\alpha}_{(\bar\sigma)\,,\bar\alpha} + \gamma^{\bar\alpha}_{\bar\alpha\bar\beta}\,a^{\bar\beta}_{(\bar\sigma)} \equiv a^{\bar\alpha}_{(\bar\sigma)\,:\bar\alpha}. \tag{5.54}$$

So the Lorentz condition can be written equivalently also as:

$$a^{\bar\alpha}_{(\bar\sigma)\,:\bar\alpha} = 0, \tag{5.55}$$

resulting fulfilled respect to both the connections Γ and γ. More, the same condition holds even in presence of non-Abelian terms, provided we replace the usual covariant derivative with the non-Abelian covariant derivative. Then it results:

$$\mathrm{D}_{\bar\alpha}\,a^{\bar\alpha}_{(\bar\sigma)} = 0, \tag{5.56}$$

being:

$$\mathrm{D}_{\bar\alpha}\,a^{\bar\alpha}_{(\bar\sigma)} \equiv a^{\bar\alpha}_{(\bar\sigma)\,:\bar\alpha} + C_{(\bar\sigma)(\bar\tau)(\bar\upsilon)}\,g^{\bar\alpha\bar\beta}\,a^{(\bar\tau)}_{\bar\alpha}\,a^{(\bar\upsilon)}_{\bar\beta}, \tag{5.57}$$

where:

$$C_{(\bar\sigma)(\bar\tau)(\bar\upsilon)}\,g^{\bar\alpha\bar\beta}\,a^{(\bar\tau)}_{\bar\alpha}\,a^{(\bar\upsilon)}_{\bar\beta} \equiv 0, \tag{5.58}$$

because of symmetries.

5.3.2 The Field Equations in the Lorentz Gauge

If we assume that the Lorentz gauge holds, the field equations (5.47) simplify considerably becoming:

$$\begin{aligned} &\widetilde{R}_{\bar{\mu}\bar{\nu}} - \tfrac{1}{2}\widetilde{R} g_{\bar{\mu}\bar{\nu}} + \mathcal{G}^{\bar{\alpha}}_{\bar{\mu}\bar{\nu}:\bar{\alpha}} - \mathcal{G}^{\bar{\alpha}}_{\bar{\mu}\bar{\beta}} \mathcal{G}^{\bar{\beta}}_{\bar{\nu}\bar{\alpha}} - \\ &-\tfrac{1}{2} g^{\gamma\delta} \Big(\mathcal{G}^{\bar{\alpha}}_{\gamma\delta:\bar{\alpha}} - \mathcal{G}^{\bar{\alpha}}_{\gamma\bar{\beta}} \mathcal{G}^{\bar{\beta}}_{\delta\bar{\alpha}} \Big) g_{\bar{\mu}\bar{\nu}} - \Lambda g_{\bar{\mu}\bar{\nu}} = 0. \end{aligned} \tag{5.59}$$

Taking into account the previous relations (5.44) and (5.45) we can write more explicitly:

$$\begin{aligned} &\widetilde{R}_{\bar{\mu}\bar{\nu}} - \tfrac{1}{2}\widetilde{R} g_{\bar{\mu}\bar{\nu}} - \tfrac{1}{2} a^{(\bar{\sigma})}_{\bar{\mu}} \Big(\mathrm{D}_{\bar{\alpha}} f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\nu}}} - \mathcal{J}_{(\bar{\sigma})\bar{\nu}} - \mathcal{A}_{(\bar{\sigma})\bar{\nu}} \Big) - \\ &-\tfrac{1}{2} \Big(\mathrm{D}_{\bar{\alpha}} f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\mu}}} - \mathcal{J}_{(\bar{\sigma})\bar{\mu}} - \mathcal{A}_{(\bar{\sigma})\bar{\mu}} \Big) a^{(\bar{\sigma})}_{\bar{\nu}} + \\ &+\tfrac{1}{2} a^{(\bar{\sigma})\bar{\beta}} \Big(\mathrm{D}_{\bar{\alpha}} f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\beta}}} - \mathcal{J}_{(\bar{\sigma})\bar{\beta}} - \mathcal{A}_{(\bar{\sigma})\bar{\beta}} \Big) g_{\bar{\mu}\bar{\nu}} - \Lambda g_{\bar{\mu}\bar{\nu}} = \\ &= f^{(\bar{\sigma})}_{\bar{\alpha}\bar{\mu}} f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\nu}}} - \tfrac{1}{4} f^{(\bar{\sigma})}_{\bar{\alpha}\bar{\beta}} f^{\bar{\alpha}\bar{\beta}}_{(\bar{\sigma})} g_{\bar{\mu}\bar{\nu}}. \end{aligned} \tag{5.60}$$

We can now introduce a new variable $\lambda_{(\bar{\sigma})\bar{\mu}}$ thanks to which we obtain the following system:

$$\begin{aligned} &\widetilde{R}_{\bar{\mu}\bar{\nu}} - \tfrac{1}{2}\widetilde{R} g_{\bar{\mu}\bar{\nu}} - a^{(\bar{\sigma})}_{\bar{\mu}} \lambda_{(\bar{\sigma})\bar{\nu}} - \lambda_{(\bar{\sigma})\bar{\mu}} a^{(\bar{\sigma})}_{\bar{\nu}} + \\ &+a^{(\bar{\sigma})\bar{\beta}} \lambda_{(\bar{\sigma})\bar{\beta}} g_{\bar{\mu}\bar{\nu}} - \Lambda g_{\bar{\mu}\bar{\nu}} = f^{(\bar{\sigma})}_{\bar{\alpha}\bar{\mu}} f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\nu}}} - \tfrac{1}{4} f^{(\bar{\sigma})}_{\bar{\alpha}\bar{\beta}} f^{\bar{\alpha}\bar{\beta}}_{(\bar{\sigma})} g_{\bar{\mu}\bar{\nu}}, \end{aligned} \tag{5.61}$$

$$\mathrm{D}_{\bar{\alpha}} f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\mu}}} - \mathcal{J}_{(\bar{\sigma})\bar{\mu}} - \mathcal{A}_{(\bar{\sigma})\bar{\mu}} = 2\lambda_{(\bar{\sigma})\bar{\mu}}, \tag{5.62}$$

which is equivalent to the previous field equations. Now:

1. if we represent $\lambda_{(\bar{\sigma})\bar{\mu}}$ on the basis of the vector potentials, as:

$$\lambda_{(\bar{\sigma})\bar{\mu}} = \lambda^{[\bar{\sigma}]} a_{(\bar{\sigma})\bar{\mu}}, \qquad \lambda^{[\bar{\sigma}]} = \lambda_{(\bar{\sigma})\bar{\mu}} a^{(\bar{\sigma})\bar{\mu}}, \tag{5.63}$$

2. we introduce the energy-momentum tensor for the fields $f^{(\bar{\sigma})}_{\bar{\mu}\bar{\nu}}$:

$$\kappa T^{[f]}_{\bar{\mu}\bar{\nu}} = f^{(\bar{\sigma})}_{\bar{\alpha}\bar{\mu}} f^{\bar{\alpha}}_{(\bar{\sigma})_{\bar{\nu}}} - \tfrac{1}{4} f^{(\bar{\sigma})}_{\bar{\alpha}\bar{\beta}} f^{\bar{\alpha}\bar{\beta}}_{(\bar{\sigma})} g_{\bar{\mu}\bar{\nu}}, \tag{5.64}$$

3. and the current density:

$$J_{(\bar{\sigma})\bar{\mu}} = \mathcal{J}_{(\bar{\sigma})\bar{\mu}} + \mathcal{A}_{(\bar{\sigma})\bar{\mu}} + \tfrac{1}{2n} f^{(\bar{\sigma})}_{\bar{\alpha}\bar{\beta}} f^{\bar{\alpha}\bar{\beta}}_{(\bar{\sigma})} g_{\bar{\mu}\bar{\nu}} + \lambda^{[\bar{\sigma}]} a_{(\bar{\sigma})\bar{\mu}}, \tag{5.65}$$

we reach a physically meaningful form of the field equations:

$$\widetilde{R}_{\bar{\mu}\bar{\nu}} - \tfrac{1}{2}\widetilde{R} g_{\bar{\mu}\bar{\nu}} - \left(\lambda^{[\bar{\sigma}]} + \Lambda\right) a_{(\bar{\sigma})\bar{\mu}}\, a^{(\bar{\sigma})}_{\bar{\nu}} = \kappa T^{[f]}_{\bar{\mu}\bar{\nu}}, \tag{5.66}$$

$$\mathrm{D}_{\bar{\alpha}} f^{\bar{\alpha}}_{(\bar{\sigma})\,\bar{\mu}} = J_{(\bar{\sigma})\bar{\mu}}. \tag{5.67}$$

Some care is required here with notations. In fact $\lambda^{[\bar{\sigma}]} a_{(\bar{\sigma})\bar{\mu}}\, a^{(\bar{\sigma})}_{\bar{\nu}}$ is equal to $\lambda^{[0]} a_{(0)\bar{\mu}}\, a^{(0)}_{\bar{\nu}} + \lambda^{[1]} a_{(1)\bar{\mu}}\, a^{(1)}_{\bar{\nu}} + \cdots + \lambda^{[n-1]} a_{(n-1)\bar{\mu}}\, a^{(n-1)}_{\bar{\nu}}$ (shortly written also as $\lambda^{[\bar{\sigma}]} g_{\bar{\mu}\bar{\nu}}$). So the new term $\lambda^{[\bar{\sigma}]}$ in (5.66) introduces an anisotropy which adds to the cosmological constant contribution.

We observe that the equation:

$$\mathrm{D}_{\bar{\alpha}} f^{\bar{\alpha}}_{(\bar{\sigma})\bar{\mu}} = J_{(\bar{\sigma})\bar{\mu}}, \tag{5.68}$$

is equivalent to:

$$\mathrm{D}_{\bar{\alpha}} \left(g^{\bar{\alpha}\bar{\beta}} f_{(\bar{\sigma})\bar{\beta}\bar{\mu}} \right) = J_{(\bar{\sigma})\bar{\mu}}. \tag{5.69}$$

Now the following relation holds between the covariant divergences of the metric tensor (remember that $g^{\bar{\alpha}\bar{\nu}}{}_{;\bar{\alpha}} = 0$ thanks to the metricity condition holding in V^n respect to the connection $\boldsymbol{\Gamma}$):

$$g^{\bar{\alpha}\bar{\nu}}{}_{;\bar{\alpha}} \equiv g^{\bar{\alpha}\bar{\nu}}{}_{:\bar{\alpha}} + \mathcal{G}^{\bar{\alpha}}_{\bar{\alpha}\bar{\beta}} g^{\bar{\beta}\bar{\nu}} + \mathcal{G}^{\bar{\nu}}_{\bar{\alpha}\bar{\beta}} g^{\bar{\alpha}\bar{\beta}} = 0, \tag{5.70}$$

which, in the Lorentz gauge, thanks to (5.31) and (5.32), ensuring that the second and third terms vanish, becomes:

$$g^{\bar{\alpha}\bar{\nu}}{}_{;\bar{\alpha}} = g^{\bar{\alpha}\bar{\nu}}{}_{:\bar{\alpha}}. \tag{5.71}$$

So, taking into account (5.41), eq (5.67) can be written also as:

$$g^{\bar{\alpha}\bar{\beta}}\, \mathrm{D}_{\bar{\alpha}} f^{(\bar{\sigma})}_{\bar{\beta}\bar{\mu}} = J^{(\bar{\sigma})}_{\bar{\mu}}. \tag{5.72}$$

Since the field strength tensor is given by:

$$f^{(\bar{\sigma})}_{\bar{\beta}\bar{\mu}} = \mathrm{D}_{\bar{\beta}}\, a^{(\bar{\sigma})}_{\bar{\mu}} - \mathrm{D}_{\bar{\mu}}\, a^{(\bar{\sigma})}_{\bar{\beta}} - C^{(\bar{\sigma})}_{\phantom{(\bar{\sigma})}(\bar{\tau})(\bar{\upsilon})} a^{(\bar{\tau})}_{\bar{\beta}} a^{(\bar{\upsilon})}_{\bar{\mu}}, \tag{5.73}$$

from (5.72) we obtain, in the Lorentz gauge, also the second order equation for the vector potentials:

$$\begin{aligned} g^{\bar{\alpha}\bar{\beta}}\, \mathrm{D}_{\bar{\alpha}}\, \mathrm{D}_{\bar{\beta}}\, a^{(\bar{\sigma})}_{\bar{\mu}} &= J^{(\bar{\sigma})}_{\bar{\mu}} + g^{\bar{\alpha}\bar{\beta}}\, \mathrm{D}_{\bar{\alpha}} \mathrm{D}_{\bar{\mu}}\, a^{(\bar{\sigma})}_{\bar{\beta}} + \\ &+ C^{(\bar{\sigma})}_{\phantom{(\bar{\sigma})}(\bar{\tau})(\bar{\upsilon})} g^{\bar{\alpha}\bar{\beta}}\, a^{(\bar{\tau})}_{\bar{\beta}}\, \mathrm{D}_{\bar{\alpha}}\, a^{(\bar{\upsilon})}_{\bar{\mu}}. \end{aligned} \tag{5.74}$$

Thanks to symmetry of the metric tensor we may exchange the order of covariant differentiation, so that it results:

$$g^{\bar{\alpha}\bar{\beta}}\, \mathrm{D}_{\bar{\alpha}} \mathrm{D}_{\bar{\mu}}\, a^{(\bar{\sigma})}_{\bar{\beta}} = g^{\bar{\alpha}\bar{\beta}}\, \mathrm{D}_{\bar{\mu}} \mathrm{D}_{\bar{\alpha}}\, a^{(\bar{\sigma})}_{\bar{\beta}} \equiv \eta^{(\bar{\sigma})(\bar{\tau})} \mathrm{D}_{\bar{\mu}} \mathrm{D}_{\bar{\alpha}}\, a^{\bar{\alpha}}_{(\bar{\tau})} = 0, \tag{5.75}$$

which is vanishing because of the Lorentz gauge condition (5.56). Then it remains:

$$g^{\bar{\alpha}\bar{\beta}}\, \mathrm{D}_{\bar{\alpha}}\, \mathrm{D}_{\bar{\beta}}\, a^{(\bar{\sigma})}_{\bar{\mu}} = J^{(\bar{\sigma})}_{\bar{\mu}} + C^{(\bar{\sigma})}_{\phantom{(\bar{\sigma})}(\bar{\tau})(\bar{\upsilon})} g^{\bar{\alpha}\bar{\beta}}\, a^{(\bar{\tau})}_{\bar{\beta}}\, \mathrm{D}_{\bar{\alpha}}\, a^{(\bar{\upsilon})}_{\bar{\mu}}. \tag{5.76}$$

The last term results to be vanishing. In fact we have:

$$\begin{aligned} C^{(\bar{\sigma})}_{\phantom{(\bar{\sigma})}(\bar{\tau})(\bar{\upsilon})} g^{\bar{\alpha}\bar{\beta}}\, a^{(\bar{\tau})}_{\bar{\beta}}\, \mathrm{D}_{\bar{\alpha}}\, a^{(\bar{\upsilon})}_{\bar{\mu}} &\equiv \mathrm{D}_{\bar{\alpha}} \left(C^{(\bar{\sigma})}_{\phantom{(\bar{\sigma})}(\bar{\tau})(\bar{\upsilon})} g^{\bar{\alpha}\bar{\beta}}\, a^{(\bar{\tau})}_{\bar{\beta}}\, a^{(\bar{\upsilon})}_{\bar{\mu}} \right) - \\ - a^{(\bar{\upsilon})}_{\bar{\mu}} C^{(\bar{\sigma})}_{\phantom{(\bar{\sigma})}(\bar{\tau})(\bar{\upsilon})} \mathrm{D}_{\bar{\alpha}} \left(g^{\bar{\alpha}\bar{\beta}}\, a^{(\bar{\tau})}_{\bar{\beta}} \right) &= 0, \end{aligned} \tag{5.77}$$

thanks to symmetries and Lorentz condition (5.56). We have finally the second order field equation:

$$g^{\bar{\alpha}\bar{\beta}}\, \mathrm{D}_{\bar{\alpha}}\, \mathrm{D}_{\bar{\beta}}\, a^{(\bar{\sigma})}_{\bar{\mu}} = J^{(\bar{\sigma})}_{\bar{\mu}}. \tag{5.78}$$

5.4 Comments and Conclusion

At the end of the chapter we want to comment especially the results (5.66), (5.67).

1. The field $f^{(\bar{\sigma})}_{\bar{\mu}\bar{\nu}}$ fulfills a Maxwellian type equation as it is expected for the non-gravitational interaction fields.

2. A new contribution $\lambda^{[\bar{\sigma}]}$ is added to the cosmological constant, representing a field which could possibly be physically interpreted as related to some contribution to *dark matter and energy.* In particular, applications to cosmology require the additional isotropy conditions $\lambda^{[0]} = \lambda^{[1]} = \cdots = \lambda^{[n]} = \lambda,$ in order to fulfill the cosmological principle (see chapter 9).

3. The energy-momentum tensor exhibits the same structure as a non -Abelian Maxwellian energy-momentum tensor. In principle one should solve, from (5.66) the metric tensor, *i.e.,* the vector potentials $a_{(\bar{\sigma})\bar{\mu}}$ as functions of the unknowns $\lambda^{[\bar{\sigma}]}$ and later solve the $\lambda^{[\bar{\sigma}]}$ from the Maxwellian equations (5.67).

Beginning from the next chapter we propose a possible physical interpretation of the mathematical theory we have sketched in the present chapter.

In particular we will be led to relate the components $a^{(\bar{\sigma})}_{\mu}, \mu = 0, 1, 2, 3$ of each vector potential $a^{(\bar{\sigma})}_{\bar{\mu}}$ to the physical fundamental interactions carried by *bosons,* while the remaining *extra* components $a^{(\bar{\sigma})}_{\underline{k}}, \underline{k} = 4, 5, \cdots, n-1$ will be related to matter fields describing *fermions.*

At the beginning of chapter 6 before all, we will examine how the fundamental fields may be confined within the physical 4-*dimensional* space-time V^4, in order that only 4 co-ordinates $x^{\mu}, \mu \ =, 1, 2, 3$ are observable. Moreover we will see how, just thanks from such a confinement process, a rest mass can arise for elementary particles, when they are observed in V^4.

Chapter 6

Interaction Fields (Bosons)

Abstract
This chapter deals at first with two non-trivial questions: i) the problem of field confinement within the observable four-dimensional space-time and ii) the matter of arising of a non-null rest mass for elementary particles. Subsequently a physical interpretation of the observable components of the eigenvectors of the metric tensor as equal to the 4-*vector* potentials of the fundamental interaction fields will be proposed.

6.1 Introduction

The mathematical theory proposed in the previous chapter, as an alternative to usual Kaluza-Klein approach to field unification, is required to pass the test of a possible and possibly non-irrelevant physical interpretation. In order to fulfill this request we proceed by two preliminary steps:

1. The former step, which will be shown, is concerned with interpreting the observable components $a_{\mu}^{(\bar{\sigma})}, (\mu = 0, 1, 2, 3)$ of the eigenvectors of the metric tensor, as the fundamental interaction fields (*gravitational, electro-weak, strong*) 4-*vector* potentials. We will see how the potentials of all the known fundamental interactions may be included into the $a_{\mu}^{(\bar{\sigma})}$, in correspondence to the different values of the label $(\bar{\sigma})$, if the dimensionality of the extended space-time V^n is $n = 16$.

2. The latter step – which will be presented in the next chapter – is involved in interpreting the *extra* components $a_{\underline{l}}^{(\bar{\sigma})}, (\underline{l} = 4, 5, \cdots, 15)$, as related to the spinor fields describing matter (*leptons, quarks*), resulting that just $n = 16$ is the number of space-time dimensions required to describe all the known fundamental matter fields.

But first of all we have to examine two more questions, *i.e.*

1. The problem of the field confinement within the physical space-time V^4;

2. The problem of non-null constant particle rest masses.

In fact, in a realistic theory, all the observable fields may depend only on the observable co-ordinates x^{μ}, *i.e.,* the fields which we have defined all over the extended space-time V^n are to be confined onto the manifold V^4. So the next section of the present chapter will be devoted to examine a possible *field confinement "mechanism",* together with a non-null particle mass constance condition.

6.2 Field Confinement and Massive Wave-Particles

The problem of field confinement and the matter of arising of non-vanishing masses for elementary particles are tightly related in our approach. As we will see they both will be solved thanks to a suitable gauge choice.

6.2.1 Vanishing of Particle Rest Masses in V^4

Let us consider the D'Alembertian field equation governing the V^n vector potential $a_{\bar{\mu}}^{(\bar{\sigma})}$ in absence of currents:

$$g^{\bar{\alpha}\bar{\beta}} a^{(\bar{\sigma})}_{\bar{\mu};\bar{\alpha};\bar{\beta}} = 0. \tag{6.1}$$

We separate now the 4-vector components in V^4:

$$g^{\bar{\alpha}\bar{\beta}} a^{(\bar{\sigma})}_{\mu;\bar{\alpha};\bar{\beta}} = 0, \qquad \mu =, 1, 2, 3, \tag{6.2}$$

which will be interpreted as related to the *physical interaction fields* carried by *bosons,* from the remaining extra components:

$$g^{\bar{\alpha}\bar{\beta}} a^{(\bar{\sigma})}_{\underline{l};\bar{\alpha};\bar{\beta}} = 0, \qquad \underline{l} = 4, 5, \cdots, n-1, \tag{6.3}$$

which behave as scalars when observed within V^4 and will be associated to the *matter fields* governing *fermions* (see chapter 7).

The previous equations can be written extensively as:

$$g^{\alpha\beta} a^{(\bar{\sigma})}_{\mu;\alpha;\beta} + g^{\underline{j}\bar{\beta}} a^{(\bar{\sigma})}_{\bar{\mu};\underline{j};\bar{\beta}} + g^{\bar{\alpha}\underline{k}} a^{(\bar{\sigma})}_{\bar{\mu};\bar{\alpha};\underline{k}} + g^{\underline{j}\underline{k}} a^{(\bar{\sigma})}_{\bar{\mu};\underline{j};\underline{k}} = 0, \tag{6.4}$$

and respectively:

$$g^{\alpha\beta} a^{(\bar{\sigma})}_{\underline{l};\alpha;\beta} + g^{\underline{j}\bar{\beta}} a^{(\bar{\sigma})}_{\underline{l};\underline{j};\bar{\beta}} + g^{\bar{\alpha}\underline{k}} a^{(\bar{\sigma})}_{\underline{l};\bar{\alpha};\underline{k}} + g^{\underline{j}\underline{k}} a^{(\bar{\sigma})}_{\underline{l};\underline{j};\underline{k}} = 0. \tag{6.5}$$

Physically we need to require covariance only respect to any transformation of the co-ordinates within V^4, which are observable, while we may accept to break covariance respect to transformations involving the *extra* co-ordinates in the entire V^n, since the latter are seen as scalars when observed from the physical space-time V^4. So distinct complex solutions will be allowed to the 4-*vector* field equations (6.4), *i.e.*:

$$a^{(\bar{\sigma})}_{\mu} \equiv c^{(\bar{\sigma})}_{\mu}\, e^{i k^{[\bar{\sigma}]}_{\alpha} x^{\alpha} + i k^{[\bar{\sigma}]}_{\underline{i}} x^{\underline{i}}}, \tag{6.6}$$

and to each of the V^4-*scalar* equations (6.5), *i.e.*:

$$a^{(\bar{\sigma})}_{\underline{l}} \equiv c^{(\bar{\sigma})}_{\underline{l}}\, e^{i k^{[\bar{\sigma}\,\underline{l}]}_{\alpha} x^{\alpha} + i k^{[\bar{\sigma}\,\underline{l}]}_{\underline{i}} x^{\underline{i}}}, \tag{6.7}$$

involving distinct wave numbers $k^{[\bar{\sigma}]}_{\alpha}$, which are the components of a 4-*vector* and $k^{[\bar{\sigma}\,\underline{l}]}_{\alpha}$ are allowed to assume different values, each for every index $\underline{l}$, related to the field $a^{(\bar{\sigma})}_{\underline{l}}$ which is seen as a scalar in 4-dimensional space-time. In correspondence to these solutions eqs (6.4) and (6.5) may be written as equivalent to Klein-Gordon equations, as:

$$g^{\alpha\beta} a^{(\bar{\sigma})}_{\mu;\alpha;\beta} + \tfrac{m^2_{[\bar{\sigma}]} c^2}{\hbar^2} a^{(\bar{\sigma})}_{\mu} = 0, \tag{6.8}$$

$$g^{\alpha\beta} a^{(\bar{\sigma})}_{\underline{l};\alpha;\beta} + \tfrac{m^2_{[\bar{\sigma}\,\underline{l}]} c^2}{\hbar^2} a^{(\bar{\sigma})}_{\underline{l}} = 0, \tag{6.9}$$

where:

$$m_{[\bar{\sigma}]} = \tfrac{\hbar}{c} \sqrt{-2 g^{\underline{j}\beta} k^{[\bar{\sigma}]}_{\underline{j}} k^{[\bar{\sigma}]}_{\beta} - g^{\underline{j}\underline{k}} k^{[\bar{\sigma}]}_{\underline{j}} k^{[\bar{\sigma}]}_{\underline{k}}}, \tag{6.10}$$

$$m_{[\bar{\sigma}\,\underline{l}]} = \tfrac{\hbar}{c} \sqrt{-2 g^{\underline{j}\beta} k^{[\bar{\sigma}\,\underline{l}]}_{\underline{j}} k^{[\bar{\sigma}\,\underline{l}]}_{\beta} - g^{\underline{j}\underline{k}} k^{[\bar{\sigma}\,\underline{l}]}_{\underline{j}} k^{[\bar{\sigma}\,\underline{l}]}_{\underline{k}}}. \tag{6.11}$$

The covariance, which is broken in V^n, but it is preserved in V^4, allows different rest mass values which will be attributed respectively to *boson* vectors carrying the fundamental interactions and to *fermions* characterizing the matter fields.

Manifestly a non vanishing wave-particle rest mass is related to the dependence of the vector potential on the extra co-ordinates $x^{\underline{i}}$. So the request that the fields $a_\mu^{(\bar{\sigma})}, a_{\underline{l}}^{(\bar{\sigma})}$ are confined within the physical space-time V^4 *i.e.,* that the latter fields may depend only on the observable co-ordinates x^α, is equivalent to require that all the particles associated to those fields has vanishing rest mass:

$$m_{[\bar{\sigma}]} = 0. \qquad m_{[\bar{\sigma}\underline{l}]} = 0. \tag{6.12}$$

6.2.2 Particle Masses Arising from Scalar Boson Gauge Fields

In order to attribute non zero masses to elementary particles we are led to consider the presence of the n-scalar gauge fields $\phi^{(\bar{\sigma})}$, which are candidate to play a role which is similar to that of the Higgs scalar boson field (see, *e.g.,* [16], [21, 22], [19]. On the CERN experiments see [1] and [12]), but here following a different "mechanism".

In fact, as we know, the n-vector potentials $a_{\bar{\mu}}^{(\bar{\sigma})}$ are determined except for a gauge function, through the gauge transformation:

$$a_{\bar{\mu}}^{(\bar{\sigma})} \;\rightarrow\; a_{\bar{\mu}}^{(\bar{\sigma})} + \phi_{;\bar{\mu}}^{(\bar{\sigma})}, \tag{6.13}$$

where the n-scalar fields $\phi^{(\bar{\sigma})}$ are required to fulfill the D'Alembertian equation:

$$g^{\bar{\alpha}\bar{\beta}}\phi_{;\bar{\alpha};\bar{\beta}}^{(\bar{\sigma})} = 0, \tag{6.14}$$

so that the Lorentz gauge is preserved. We emphasize that those scalar fields do not appear into the observable tensors $f_{\bar{\mu}\bar{\nu}}^{(\bar{\sigma})}$ which are gauge invariant, since:

$$f^{(\bar\sigma)}_{\bar\mu\bar\nu} = a^{(\bar\sigma)}_{\bar\nu;\bar\mu} - a^{(\bar\sigma)}_{\bar\mu;\bar\nu} \equiv a^{(\bar\sigma)}_{\bar\nu,\bar\mu} - a^{(\bar\sigma)}_{\bar\mu,\bar\nu} \equiv \left(a^{(\bar\sigma)}_{\bar\nu,\bar\mu} + \phi^{(\bar\sigma)}_{,\bar\nu,\bar\mu}\right) - \left(a^{(\bar\sigma)}_{\bar\mu,\bar\nu} + \phi^{(\bar\sigma)}_{,\bar\mu,\bar\nu}\right).$$

Then the dependence of $\phi^{(\bar\sigma)}$ on the extra co-ordinates $x^{\underline{l}}$ does not affect the observables $f^{(\bar\sigma)}_{\bar\mu\bar\nu}$ and cannot be detected by direct observation within the physical space-time V^4. Introducing now a wave solution for $\phi^{(\bar\sigma)}$ like:

$$\phi^{(\bar\sigma)} \equiv C^{(\bar\sigma)}\, e^{iK^{[\bar\sigma]}_{\alpha}x^{\alpha} + iK^{[\bar\sigma]}_{\underline{i}}x^{\underline{i}}}, \tag{6.15}$$

into (6.14) we obtain the Klein-Gordon equation:

$$g^{\alpha\beta}\phi^{(\bar\sigma)}_{;\alpha;\beta} + \tfrac{M^2_{[\bar\sigma]}c^2}{\hbar^2}\phi^{(\bar\sigma)} = 0. \tag{6.16}$$

The rest mass of the scalar boson associated to the field $\phi^{(\bar\sigma)}$ is just:

$$M_{[\bar\sigma]} = \tfrac{\hbar}{c}\sqrt{-2g^{\underline{j}\beta}K^{[\bar\sigma]}_{\underline{j}}K^{[\bar\sigma]}_{\beta} - g^{\underline{j}\underline{k}}K^{[\bar\sigma]}_{\underline{j}}K^{[\bar\sigma]}_{\underline{k}}}. \tag{6.17}$$

We remark that while a single solution $\phi^{(\bar\sigma)}$ is required for the covariance of the 4-*vector* $\phi^{(\bar\sigma)}_{;\alpha}$ in V^4, different solutions:

$$\phi^{(\bar\sigma\underline{l})} \equiv C^{(\bar\sigma\underline{l})}\, e^{iK^{[\bar\sigma\underline{l}]}_{\alpha}x^{\alpha} + iK^{[\bar\sigma\underline{l}]}_{\underline{i}}x^{\underline{i}}}, \tag{6.18}$$

are allowed for each value of the index $\underline{l}$, to obtain *extra* components $\phi^{(\bar\sigma)}_{;\underline{l}}$, each of them being a scalar respect to transformations of the co-ordinates x^{α} within V^4. So different masses:

$$M_{[\bar\sigma\underline{l}]} = \tfrac{\hbar}{c}\sqrt{-2g^{\underline{j}\beta}K^{[\bar\sigma\underline{l}]}_{\underline{j}}K^{[\bar\sigma\underline{l}]}_{\beta} - g^{\underline{j}\underline{k}}K^{[\bar\sigma\underline{l}]}_{\underline{j}}K^{[\bar\sigma\bar{l}]}_{\underline{k}}}, \tag{6.19}$$

arise for the scalar fields contributing respectively to *bosons* $\left(M_{[\bar\sigma]}\right)$ and *fermions* $\left(M_{[\bar\sigma\underline{l}]}\right)$.

If we assume that the $a^{(\bar\sigma)}_{\mu}$ depend only on x^{α} (so that $m_{[\bar\sigma]} = 0$), while $\phi^{(\bar\sigma)}$ may depend also on $x^{\underline{l}}$, then the gauge transformation (6.13) implies into (6.8):

$$g^{\alpha\beta}a^{(\bar{\sigma})}_{\mu;\alpha;\beta} + g^{\alpha\beta}\phi^{(\bar{\sigma})}_{;\mu;\alpha;\beta} + \tfrac{M^2_{[\bar{\sigma}]}c^2}{\hbar^2}\phi^{(\bar{\sigma})}_{;\mu} = 0, \tag{6.20}$$

which in correspondence to the solution (6.15) for the field $\phi^{(\bar{\sigma})}$ may be written as:

$$g^{\alpha\beta}a^{(\bar{\sigma})}_{\mu;\alpha;\beta} + i\,K^{[\bar{\sigma}]}_{\mu}\Big(g^{\alpha\beta}\phi^{(\bar{\sigma})}_{;\alpha;\beta} + \tfrac{M^2_{[\bar{\sigma}]}c^2}{\hbar^2}\phi^{(\bar{\sigma})}\Big) = 0. \tag{6.21}$$

Thanks to (6.16) the gauge invariance is preserved in V^4, resulting:

$$g^{\alpha\beta}a^{(\bar{\sigma})}_{\mu;\alpha;\beta} = 0, \tag{6.22}$$

Let us now consider the following transformation of field variables:

$$\begin{aligned} \widehat{a}^{(\bar{\sigma})}_{\bar{\mu}} &= \tfrac{1}{\sqrt{2}}\Big(a^{(\bar{\sigma})}_{\bar{\mu}} + i\,K^{[\bar{\sigma}]}_{\bar{\mu}}\phi^{(\bar{\sigma})}\Big), \\ i\,\widehat{K}^{[\bar{\sigma}]}_{\mu}\widehat{\phi}^{(\bar{\sigma})} &= \tfrac{1}{\sqrt{2}}\Big(i\,K^{[\bar{\sigma}]}_{\bar{\mu}}\phi^{(\bar{\sigma})} - a^{(\bar{\sigma})}_{\bar{\mu}}\Big), \end{aligned} \tag{6.23}$$

and its inverse:

$$\begin{aligned} a^{(\bar{\sigma})}_{\bar{\mu}} &= \tfrac{1}{\sqrt{2}}\Big(\widehat{a}^{(\bar{\sigma})}_{\bar{\mu}} - i\,\widehat{K}^{[\bar{\sigma}]}_{\mu}\widehat{\phi}^{(\bar{\sigma})}\Big), \\ i\,K^{[\bar{\sigma}]}_{\bar{\mu}}\phi^{(\bar{\sigma})} &= \tfrac{1}{\sqrt{2}}\Big(i\,\widehat{K}^{[\bar{\sigma}]}_{\mu}\widehat{\phi}^{(\bar{\sigma})} + \widehat{a}^{(\bar{\sigma})}_{\bar{\mu}}\Big). \end{aligned} \tag{6.24}$$

In fact one is able to verify by direct computation that considering a transformation law $\boldsymbol{w} = \boldsymbol{A}\boldsymbol{v}$, where:

$$\widehat{\boldsymbol{v}} \equiv \begin{pmatrix} \widehat{a}^{(\bar{\sigma})}_{\bar{\mu}} \\ \\ i\,\widehat{K}^{[\bar{\sigma}]}_{\mu}\widehat{\phi}^{(\bar{\sigma})} \end{pmatrix}, \quad \boldsymbol{A} \equiv \tfrac{1}{\sqrt{2}}\begin{pmatrix} \delta^{\bar{\nu}}_{\bar{\mu}} & \delta^{\bar{\nu}}_{\bar{\mu}} \\ \\ -\delta^{\bar{\nu}}_{\bar{\mu}} & \delta^{\bar{\nu}}_{\bar{\mu}} \end{pmatrix}, \quad \boldsymbol{v} \equiv \begin{pmatrix} a^{(\bar{\sigma})}_{\bar{\mu}} \\ \\ i\,K^{[\bar{\sigma}]}_{\mu}\phi^{(\bar{\sigma})} \end{pmatrix}.$$

the inverse transformation $\boldsymbol{v} = \boldsymbol{A}^{-1}\widehat{\boldsymbol{v}}$, has a coefficient matrix:

$$\boldsymbol{A}^{-1} \equiv \tfrac{1}{\sqrt{2}}\begin{pmatrix} \delta^{\bar{\nu}}_{\bar{\mu}} & -\delta^{\bar{\nu}}_{\bar{\mu}} \\ \\ \delta^{\bar{\nu}}_{\bar{\mu}} & \delta^{\bar{\nu}}_{\bar{\mu}} \end{pmatrix},$$

for which it results:

$$\mathsf{A}^{-1}\mathsf{A} \equiv \tfrac{1}{2}\begin{pmatrix} \delta^{\bar\nu}_{\bar\mu} & \delta^{\bar\nu}_{\bar\mu} \\ & \\ -\delta^{\bar\nu}_{\bar\mu} & \delta^{\bar\nu}_{\bar\mu} \end{pmatrix}\begin{pmatrix} \delta^{\bar\rho}_{\bar\nu} & -\delta^{\bar\rho}_{\bar\nu} \\ & \\ \delta^{\bar\rho}_{\bar\nu} & \delta^{\bar\rho}_{\bar\nu} \end{pmatrix} \equiv \begin{pmatrix} \delta^{\bar\rho}_{\bar\mu} & 0 \\ & \\ 0 & \delta^{\bar\rho}_{\bar\mu} \end{pmatrix} \equiv \mathsf{I}.$$

Now:

1. Substitution of (6.24) into (6.22) leads to:

$$g^{\alpha\beta}\,\widehat{a}^{(\bar\sigma)}_{\bar\mu\,;\alpha;\beta} - i\,\widehat{K}^{[\bar\sigma]}_{\mu}\,g^{\alpha\beta}\,\widehat{\phi}^{(\bar\sigma)}_{;\alpha;\beta} = 0. \tag{6.25}$$

2. Substitution of (6.24) into (6.16) leads to:

$$g^{\alpha\beta}\left(\widehat{a}^{(\bar\sigma)}_{\bar\mu} + i\,\widehat{K}^{[\bar\sigma]}_{\mu}\,\widehat{\phi}^{(\bar\sigma)}\right)_{;\alpha;\beta} + \frac{M^2_{[\bar\sigma]}c^2}{\hbar^2}\left(\widehat{a}^{(\bar\sigma)}_{\bar\mu} + i\,\widehat{K}^{[\bar\sigma]}_{\mu}\,\widehat{\phi}^{(\bar\sigma)}\right) = 0. \tag{6.26}$$

And then:

$$g^{\alpha\beta}\,\widehat{a}^{(\bar\sigma)}_{\bar\mu\,;\alpha;\beta} + \frac{M^2_{[\bar\sigma]}c^2}{\hbar^2}\widehat{a}^{(\bar\sigma)}_{\bar\mu} + i\,\widehat{K}^{[\bar\sigma]}_{\mu}\left(g^{\alpha\beta}\,\widehat{\phi}^{(\bar\sigma)}_{;\alpha;\beta} + \frac{M^2_{[\bar\sigma]}c^2}{\hbar^2}\,\widehat{\phi}^{(\bar\sigma)}\right) = 0. \tag{6.27}$$

After the transformation the fields $\widehat{a}^{(\bar\sigma)}_{\bar\mu}, \widehat{\phi}^{(\bar\sigma)}$ are required to fulfill the Klein-Gordon equations:

$$g^{\alpha\beta}\,\widehat{a}^{(\bar\sigma)}_{\bar\mu\,;\alpha;\beta} + \frac{\widehat{m}^2_{[\bar\sigma]}c^2}{\hbar^2}\widehat{a}^{(\bar\sigma)}_{\bar\mu} = 0, \tag{6.28}$$

$$g^{\alpha\beta}\,\widehat{\phi}^{(\bar\sigma)}_{;\alpha;\beta} + \frac{\widehat{M}^2_{[\bar\sigma]}c^2}{\hbar^2}\,\widehat{\phi}^{(\bar\sigma)} = 0, \tag{6.29}$$

where the new particle rest masses, denoted as $\widehat{m}_{[\bar\sigma]}, \widehat{M}_{[\bar\sigma]}$, are to be determined. Combining (6.28) and (6.29) with (6.25) and (6.27) we obtain:

$$\left(\widehat{m}^2_{[\bar\sigma]} - M^2_{[\bar\sigma]}\right)\widehat{a}^{(\bar\sigma)}_{\bar\mu} = 0, \tag{6.30}$$

$$\left(\widehat{m}^2_{[\bar\sigma]} - M^2_{[\bar\sigma]}\right)\widehat{a}^{(\bar\sigma)}_{\bar\mu} + \left(\widehat{M}^2_{[\bar\sigma]} - M^2_{[\bar\sigma]}\right)i\,\widehat{K}^{[\bar\sigma]}_{\mu}\,\widehat{\phi}^{(\bar\sigma)} = 0. \tag{6.31}$$

Then the relations between the masses results to be:

$$\widehat{m}_{[\bar{\sigma}]} = M_{[\bar{\sigma}]}, \qquad \widehat{M}_{[\bar{\sigma}]} = M_{[\bar{\sigma}]}. \tag{6.32}$$

Therefore non vanishing particle rest masses for the vector bosons carrying the interaction fields arise, resulting into (6.28):

$$g^{\alpha\beta}\, \widehat{a}^{(\bar{\sigma})}_{\bar{\mu};\alpha;\beta} + \frac{M^2_{[\bar{\sigma}]}c^2}{\hbar^2} \widehat{a}^{(\bar{\sigma})}_{\bar{\mu}} = 0, \tag{6.33}$$

and into (6.29):

$$g^{\alpha\beta}\, \widehat{\phi}^{(\bar{\sigma})}_{;\alpha;\beta} + \frac{M^2_{[\bar{\sigma}]}c^2}{\hbar^2} \widehat{\phi}^{(\bar{\sigma})} = 0. \tag{6.34}$$

According to the previous approach it results that non-vanishing masses of interaction vector bosons arise and are equal to the scalar boson masses, which are hidden within the *extra* dimensions of space-time.

We observe that any further gauge transformation: $a^{(\bar{\sigma})}_{\mu} \to a^{(\bar{\sigma})}_{\mu} + f^{(\bar{\sigma})}_{;\mu}(x^{\alpha})$ is always possible involving only the observable co-ordinates x^{α}, while the gauge is fixed only in the *extra* space-time.

Of course the same procedure can be implemented also respect to the *extra* components $a^{(\bar{\sigma})}_{\underline{l}}$, obtaining the rest masses:

$$m_{[\bar{\sigma}\underline{l}]} = M_{[\bar{\sigma}\underline{l}]}, \tag{6.35}$$

which will be related to the fermions (*leptons* and *quarks*) as we will show in the next chapter.

We conclude the section observing that the gauge fields $\phi^{(\bar{\sigma})}$ may be thought also related to only one scalar field $\varPhi$, according to the relation:

$$\phi^{(\bar{\sigma})} = \delta^{(\bar{\sigma})}_{[\bar{\tau}]}\, \varPhi^{\, g^{[\bar{\tau}]}}, \tag{6.36}$$

where the exponents $g^{[\bar{\tau}]}$ are coupling constants of the vector potentials with a single scalar boson field:

$$\varPhi = C\, e^{iK_{\alpha}x^{\alpha} + iK_{\underline{i}}x^{\underline{i}}}, \tag{6.37}$$

which fulfills the Klein-Gordon field equation:

$$g^{\alpha\beta}\Phi_{;\alpha;\beta} + \frac{M^2c^2}{\hbar^2}\Phi = 0. \tag{6.38}$$

Then it results:

$$\phi^{(\bar{\sigma})} = \delta^{(\bar{\sigma})}_{[\bar{\tau}]} C^{g^{[\bar{\tau}]}} e^{g^{[\bar{\tau}]}\left(iK_\alpha x^\alpha + iK_{\underline{i}} x^{\underline{i}}\right)}, \tag{6.39}$$

and:

$$\phi^{(\bar{\sigma})}_{;\bar{\mu}} = i\,K_{\bar{\mu}}\, g^{[\bar{\tau}]}\, \delta^{(\bar{\sigma})}_{[\bar{\tau}]}\, \phi^{(\bar{\sigma})}. \tag{6.40}$$

Since $\phi^{(\bar{\sigma})}_{;\bar{\mu}} = K^{[\bar{\sigma}]}_{\bar{\mu}}\, \phi^{(\bar{\sigma})}$, we obtain also:

$$K^{[\bar{\sigma}]}_{\bar{\mu}} = g^{[\bar{\sigma}]}\, K_{\bar{\mu}}. \tag{6.41}$$

Then the masses of the vector bosons carrying the interaction fields would become:

$$\widehat{m}_{[\bar{\sigma}]} = g^{[\bar{\sigma}]}\, M, \tag{6.42}$$

being:

$$M = \frac{\hbar}{c}\sqrt{-2g^{\underline{j}\beta}K_{\underline{j}}K_\beta - g^{\underline{j}\underline{k}}K_{\underline{j}}K_{\underline{k}}}. \tag{6.43}$$

The values of $\widehat{m}_{[\bar{\sigma}]}$ depend on the coupling constant with the scalar boson of mass M. Similarly the masses of fermions will result:

$$\widehat{m}_{[\bar{\sigma}\underline{l}]} = g^{[\bar{\sigma}\underline{l}]}\, M, \tag{6.44}$$

being:

$$\phi^{(\bar{\sigma}\underline{l})} = \delta^{(\bar{\sigma})}_{[\bar{\tau}]}\, \Phi^{\, g^{[\bar{\tau}\underline{l}]}}, \tag{6.45}$$

and $g^{[\bar{\tau}\underline{l}]}$ the coupling constants of the fields $a^{(\bar{\sigma})}_{\underline{l}}$ with the scalar field Φ.

6.3 The Fundamental Interaction Fields

After having examined the preliminary matters of the relation between field confinement and non-vanishing rest masses of elementary particles, we consider the subject of the physical interpretation of the fields $a_\mu^{(\bar{\sigma})}$.

6.3.1 The Gravitational Field

We start considering the gravitational fields in ordinary space-time.

Case $n = 4$

The identification of the vector potential components which are suitable to describe the gravitational field in the physical space-time V^4, is easily suggested by the consideration of the theory in the special situation when the dimensionality of the entire space-time is just $n = 4$. In this occurrence the metric tensor is given by:

$$g_{\mu\nu} = \eta_{(\sigma)(\tau)} a_\mu^{(\sigma)} a_\nu^{(\tau)}, \qquad \mu, \nu, \sigma, \tau = 0, 1, 2, 3, \tag{6.46}$$

and the potentials $a_\mu^{(\sigma)}$ are clearly responsible of gravitation, according to general relativity. The connection coefficient becomes now:

$$\Gamma^\alpha_{\mu\nu} = \gamma^\alpha_{\mu\nu} - \tfrac{1}{2}\left(a_\mu^{(\sigma)} f^\alpha_{(\sigma)\nu} + a_\nu^{(\sigma)} f^\alpha_{(\sigma)\mu} \right), \tag{6.47}$$

where:

$$\gamma^\alpha_{\mu\nu} = \tfrac{1}{2} a_\mu^{(\sigma)} h^\alpha_{(\sigma)\nu}, \tag{6.48}$$

are the coefficients of the *reduced connection* γ. The Ricci tensor, when written in terms of the *full* connection $\boldsymbol{\Gamma}$ has simply the usual form:

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} + \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta}. \tag{6.49}$$

So the Einstein equations are the as usual:

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = 0, \tag{6.50}$$

where no energy-momentum tensor appears, since the whole gravitational field is included into space-time geometry.

But if we represent the Ricci tensor in terms of the reduced connection γ, which only partially includes the gravitational field into geometry and leaves part of it as an external field of strength $\boldsymbol{f}^{(\sigma)} \equiv (f^{(\sigma)}_{\alpha\beta})$ we have the field equations, in the Lorentz gauge (see §5.3.2):

$$\widetilde{R}_{\mu\nu} - \tfrac{1}{2} \widetilde{R} g_{\mu\nu} - \Lambda g_{\mu\nu} - \lambda^{[\sigma]} a_{(\sigma)\mu} a^{(\sigma)}_{\nu} = \kappa T^{[g]}_{\mu\nu}, \tag{6.51}$$

$$\mathrm{D}_\alpha f^{\alpha}_{(\sigma)_\mu} = J_{(\sigma)\mu}, \tag{6.52}$$

in the r.h.s. of which the following energy-momentum tensor of the (non-embedded into geometry) contribution to the gravitational field:

$$\kappa T^{[g]}_{\mu\nu} = f^{(\sigma)}_{\alpha\mu} f^{\alpha}_{(\sigma)_\nu} - \tfrac{1}{4} f^{(\sigma)}_{\alpha\beta} f^{\alpha\beta}_{(\sigma)} g_{\mu\nu}, \tag{6.53}$$

and the gravitational current density:

$$J_{(\sigma)\mu} = \mathcal{J}_{(\sigma)\mu} + \mathcal{A}_{(\sigma)\mu} + 2\lambda^{[\sigma]} a_{(\sigma)\mu}, \tag{6.54}$$

appear. We observe that even the l.h.s. of (6.51) might be thought of as equivalent to an energy-momentum contribution hidden into geometry:

$$\kappa T^{[G]}_{\mu\nu} = -\widetilde{R}_{\mu\nu} + \tfrac{1}{2} \widetilde{R} g_{\mu\nu} - \Lambda g_{\mu\nu} - \lambda^{[\sigma]} a_{(\sigma)\mu} a^{(\sigma)}_{\nu}. \tag{6.55}$$

So the field equations for the gravitational field could be written equivalently in the *energetic form*:

$$T^{[gr]}_{\mu\nu} = 0, \tag{6.56}$$

where:

$$T^{[gr]}_{\mu\nu} = T^{[G]}_{\mu\nu} + T^{[g]}_{\mu\nu}. \tag{6.57}$$

As we will see in chapter 9 the reduction of the Einstein equation in terms of energy-momentum tensor will be relevant in order to quantization of the gravitational field.

Case $n > 4$

We are naturally led to continue to associate the potentials labeled by $\sigma = 0, 1, 2, 3,$ to the gravitational field even when the space time dimensionality is greater than 4, while the components of additional potentials $a^{(\underline{s})}_{\bar{\mu}}$, $\underline{s} = 4, 5, \cdots, n-1$, and $\bar{\mu} = 0, 1, 2, 3, \cdots n-1$, will be related to non-gravitational interaction fields.

We observe that when $n > 4$ each vector potential involves also additional components $a^{(\bar{\sigma})}_{\underline{l}}, \underline{l} = 4, 5, \cdots, n-1$, because of the increased dimensionality of space-time.

A compact decomposition of the potentials may result to be useful:

$$a^{(\bar{\sigma})}_{\bar{\mu}} = \delta^{(\bar{\sigma})}_{\sigma}\left(\delta^{\mu}_{\bar{\mu}}\, a^{(\sigma)}_{\mu} + \delta^{l}_{\bar{\mu}}\, a^{(\sigma)}_{\underline{l}}\right) + \delta^{(\bar{\sigma})}_{\underline{s}}\left(\delta^{\mu}_{\bar{\mu}}\, a^{(\underline{s})}_{\mu} + \delta^{l}_{\bar{\mu}}\, a^{(\underline{s})}_{\underline{l}}\right). \tag{6.58}$$

Now we are led to the following physical interpretation.

1. The *gravitational* field observed within the physical space-time V^4 is now represented by the $a^{(\sigma)}_{\mu}$, $(\sigma, \mu = 0, 1, 2, 3)$, which are four-vectors in V^4.
2. The *electro-weak* and *strong* interaction fields are associated to the $a^{(\underline{s})}_{\mu}$, ($\underline{s} = 4, 5, \cdots, n-1$), which are also four-vectors in V^4.
3. The remaining components $a^{(\bar{\sigma})}_{\underline{l}} = \delta^{(\bar{\sigma})}_{\sigma}\, a^{(\sigma)}_{\underline{l}} + \delta^{(\bar{\sigma})}_{(\underline{s})}\, a^{(\underline{s})}_{\underline{l}}$, which behave as scalars within the observable space-time V^4 are related to the matter fields, *i.e.,* to the fundamental fermions (*leptons, quarks*).

We observe also that the sectors of the potentials of indices $\mu, \underline{l}$:

$$
\begin{aligned}
a_{\mu}^{(\sigma)} &= c_{\mu}^{(\sigma)}\, e^{ik_{\alpha}^{[\sigma]}x^{\alpha}+ik_{\underline{i}}^{[\sigma]}x^{\underline{i}}}, \qquad & a_{\mu}^{(\underline{s})} &= c_{\mu}^{(\underline{s})}\, e^{ik_{\alpha}^{[\underline{s}]}x^{\alpha}+ik_{\underline{i}}^{[\underline{s}]}x^{\underline{i}}}, \\
a_{\underline{l}}^{(\sigma)} &= c_{\underline{l}}^{(\sigma)}\, e^{ik_{\alpha}^{[\sigma,\underline{l}]}x^{\alpha}+ik_{\underline{i}}^{[\sigma,\underline{l}]}x^{\underline{i}}}, \qquad & a_{\underline{l}}^{(\underline{s})} &= c_{\underline{l}}^{(\underline{s})}\, e^{ik_{\alpha}^{[\underline{s},\underline{l}]}x^{\alpha}+ik_{\underline{i}}^{[\underline{s},\underline{l}]}x^{\underline{i}}},
\end{aligned}
\tag{6.59}
$$

solutions to the corresponding Klein-Gordon field equations:

$$
\begin{aligned}
g^{\alpha\beta}a_{\mu;\alpha;\beta}^{(\sigma)} + \frac{m_{[\sigma]}^{2}c^{2}}{\hbar^{2}}a_{\mu}^{(\sigma)} &= 0, \qquad & g^{\alpha\beta}a_{\mu;\alpha;\beta}^{(\underline{s})} + \frac{m_{[\underline{s}]}^{2}c^{2}}{\hbar^{2}}a_{\mu}^{(\underline{s})} &= 0, \\
g^{\alpha\beta}a_{\underline{l};\alpha;\beta}^{(\sigma)} + \frac{m_{[\sigma,\underline{l}]}^{2}c^{2}}{\hbar^{2}}a_{\underline{l}}^{(\sigma)} &= 0, \qquad & g^{\alpha\beta}a_{\underline{l};\alpha;\beta}^{(\underline{s})} + \frac{m_{[\underline{s},\underline{l}]}^{2}c^{2}}{\hbar^{2}}a_{\underline{l}}^{(\underline{s})} &= 0,
\end{aligned}
\tag{6.60}
$$

may be associated to particles of different rest masses:

$$
m_{[\sigma]} = \frac{\hbar}{c}\sqrt{k_{\alpha}^{[\sigma]}k_{[\sigma]}^{\alpha}}, \qquad m_{[\underline{s},\underline{l}]} = \frac{\hbar}{c}\sqrt{k_{\alpha}^{[\underline{s},\underline{l}]}k_{[\underline{s},\underline{l}]}^{\alpha}}, \tag{6.61}
$$

if we allow that the covariance is preserved only in the observable space-time V^4, while it may be broken in the *extra* dimensions of the multidimensional space-time V^n. In fact each field component $a_{\underline{l}}^{(\sigma)}, a_{\underline{l}}^{(\underline{s})}$ is a scalar respect to co-ordinate transformation affecting only V^4 and fulfills its own Klein-Gordon equation. This circumstance will be useful when dealing with fermions, the field spinor components of which will be associated to the *extra* components $a_{\underline{l}}^{(\bar{\sigma})}, a_{\underline{l}}^{(\underline{s})}$.

We emphasize that all the previous masses $m_{[\sigma]}, m_{[\underline{s},\underline{l}]}$ result to be null when the vector fields depend only on x^{α}. So gravitons have vanishing masses. While the non zero rest masses of fermions associated to the $a_{\underline{l}}^{(\sigma)}$ arise thanks to the contribution of the gauge fields $\phi_{;\underline{l}}^{(\sigma)}$ to their masses.

The field equations for the gravitational field in V^4, respect to eq (6.51), involve, beside the terms labelled by $\sigma = 0, 1, 2, 3$, the new contributions labelled by $\underline{l} = 4, 5, \cdots, n-1$, and the correspondent additional energy-momentum tensor term. So we have:

$$\begin{aligned}
&\widetilde{R}_{\mu\nu} - \tfrac{1}{2}\widetilde{R}g_{\mu\nu} - \Lambda g_{\mu\nu} - \lambda^{[\sigma]} a_{(\sigma)\mu} a_{\nu}^{(\sigma)} - \\
&-\lambda^{[\underline{s}]} a_{(\underline{s})\mu} a_{\nu}^{(\underline{s})} = \kappa T_{\mu\nu}^{[g]} + \kappa T_{\mu\nu}^{[f]},
\end{aligned} \tag{6.62}$$

$$\mathrm{D}_{\alpha} f_{(\sigma)_{\mu}}^{\alpha} = J_{(\sigma)\mu}, \tag{6.63}$$

where:

$$\kappa T^{[f]} = f_{\alpha\mu}^{(\underline{s})} f_{(\underline{s})_{\nu}}^{\alpha} - \tfrac{1}{4} f_{\alpha\beta}^{(\underline{s})} f_{(\underline{s})}^{\alpha\beta} g_{\mu\nu}, \tag{6.64}$$

is the energy-momentum of the non-gravitational Maxwellian fields as they may be observed within the physical space-time V^4.

The residual equations for the *extra* terms of potentials do not involve the gravitational field but the non-gravitational *boson electro-weak, strong* $\left(a_{\mu}^{(\underline{s})}\right)$ and the *fermion* matter fields $\left(a_{\underline{l}}^{(\sigma)}, a_{\underline{l}}^{(\underline{s})}\right)$. A more familiar form of the Einstein equations, which hides the whole gravitational field into geometry is obtained if we write the metric tensor as:

$$g_{\bar{\mu}\bar{\nu}} = \bar{g}_{\bar{\mu}\bar{\nu}} + a_{\bar{\mu}}^{(\underline{s})} a_{(\underline{s})\bar{\nu}}, \tag{6.65}$$

where:

$$\bar{g}_{\bar{\mu}\bar{\nu}} = a_{\bar{\mu}}^{(\sigma)} a_{(\sigma)\bar{\nu}}, \tag{6.66}$$

includes only the gravitational vector potentials $a_{\bar{\mu}}^{(\sigma)}, \sigma = 0, 1, 2, 3$.

The connection coefficients write now as:

$$\Gamma_{\bar{\mu}\bar{\nu}}^{\bar{\alpha}} = \overline{\gamma}_{\bar{\mu}\bar{\nu}}^{\bar{\alpha}} - \tfrac{1}{2}\left(a_{\bar{\mu}}^{(\underline{s})} f_{(\underline{s})_{\bar{\nu}}}^{\bar{\alpha}} + f_{(\underline{s})_{\bar{\mu}}}^{\bar{\alpha}} a_{\bar{\nu}}^{(\underline{s})}\right), \tag{6.67}$$

where the coefficients of the new *partially reduced connection* $\overline{\gamma}$ are defined as:

$$\overline{\gamma}_{\bar{\mu}\bar{\nu}}^{\bar{\alpha}} = \gamma_{\bar{\mu}\bar{\nu}}^{\bar{\alpha}} - \tfrac{1}{2}\left(a_{\bar{\mu}}^{(\sigma)} f_{(\sigma)_{\bar{\nu}}}^{\bar{\alpha}} + f_{(\sigma)_{\bar{\mu}}}^{\bar{\alpha}} a_{\bar{\nu}}^{(\sigma)}\right). \tag{6.68}$$

In this way the gravitational field is entirely hidden into geometry and only the non-gravitational fields contribute to the energy-momentum tensor.

In fact now the field equations become:

$$\overline{R}_{\mu\nu} - \tfrac{1}{2}\overline{R}\,g_{\mu\nu} - \Lambda g_{\mu\nu} - \lambda^{[\underline{s}]}a_{(\underline{s})\mu}a_{\nu}^{(\underline{s})} = \kappa T_{\mu\nu}^{[f]}, \tag{6.69}$$

in which:

$$\overline{R}_{\mu\nu} = \overline{\gamma}^{\bar{\alpha}}_{\mu\nu,\bar{\alpha}} - \overline{\gamma}^{\bar{\alpha}}_{\mu\bar{\alpha},\nu} - \overline{\gamma}^{\bar{\alpha}}_{\mu\bar{\beta}}\overline{\gamma}^{\bar{\beta}}_{\nu\bar{\alpha}} + \overline{\gamma}^{\bar{\alpha}}_{\mu\nu}\overline{\gamma}^{\bar{\beta}}_{\bar{\alpha}\bar{\beta}}, \tag{6.70}$$

are the V^4 components of the new *partially reduced Ricci tensor* $\overline{R}$ evaluated respect to the *partially reduced connection* $\overline{\gamma}$. Of course, now, the Maxwellian equations for the gravitational vector potentials $a_{\mu}^{(\sigma)}$ disappear since the latter potentials are now absorbed into $\bar{g}_{\mu\nu}$.

Respect to the expected Einstein equations in V^4 a new term has appeared, *i.e.*:

$$L_{\mu\nu}^{[d]} = \lambda^{[\underline{s}]}a_{(\underline{s})\mu}a_{\nu}^{(\underline{s})}. \tag{6.71}$$

Moreover $\overline{R}_{\mu\nu}$ differs from the expected Ricci tensor $R_{\mu\nu}^{<4>}$ in V^4, being evaluated respect to $\overline{\gamma}^{\bar{\alpha}}_{\mu\nu}$ instead of:

$$\Gamma^{\alpha}_{<4>\mu\nu} = \tfrac{1}{2}g^{\alpha\beta}_{<4>}\left(g^{<4>}_{\mu\beta,\nu} + g^{<4>}_{\nu\beta,\mu} - g^{<4>}_{\mu\nu,\beta}\right), \quad g_{\mu\nu}^{<4>} = a_{\mu}^{(\sigma)}a_{(\sigma)\nu}. \tag{6.72}$$

Then a complete comparison between the usual Einstein equations in V^4 and (6.51) is possible if we write:

$$R_{\mu\nu}^{<4>} - \tfrac{1}{2}R^{<4>}g_{\mu\nu} + L_{\mu\nu}^{[d]} = \kappa T_{\mu\nu}^{[f]} + \kappa T_{\mu\nu}^{[d]}, \tag{6.73}$$

where:

$$R_{\mu\nu}^{<4>} = \Gamma^{\alpha}_{<4>\mu\nu,\alpha} - \Gamma^{\alpha}_{<4>\mu\alpha,\nu} - \Gamma^{\alpha}_{<4>\mu\beta}\Gamma^{\beta}_{<4>\nu\alpha} + \Gamma^{\alpha}_{<4>\mu\nu}\Gamma^{\beta}_{<4>\alpha\beta}, \tag{6.74}$$

is the usual Ricci tensor in V^4 and:

$$\kappa T_{\mu\nu}^{[d]} = R_{\mu\nu}^{<4>} - \tfrac{1}{2}R^{<4>}g_{\mu\nu} - \overline{R}_{\mu\nu} + \tfrac{1}{2}\overline{R}\,g_{\mu\nu}. \tag{6.75}$$

The additional contributions $L_{\mu\nu}^{[d]}$ and $T_{\mu\nu}^{[d]}$ of the energy-momentum tensor, appear to be responsible of *dark energy* and *dark matter* emergence, but as we will see in chapter 8, dealing with cosmology, they vanish if the metric tensor is diagonal. So small anisotropies, in universe, near galaxies could play an important role.

6.3.2 The Electro-Weak Interaction Field

The *extra* space dimensions have been introduced in order to obtain non-vanishing particle rest masses and to allow the presence of non-gravitational fields, *i.e., electro-weak* and *strong* interaction fields. Of course it is irrelevant the order according to which these fields are associated to each of the remaining vector potentials $a_\mu^{(\underline{s})}, \underline{s} = 4, 5, \cdots, n-1$, since the labels can be always renamed. So we can state to start, *e.g.,* with the *electro-weak* interactions which are carried by *massless photons,* and $W^\pm, Z^0$ *massive bosons.* In order to describe unified electromagnetic and weak fields we need one Abelian field and three *non*-Abelian ones. So the indices $\underline{s} = 4, 5, 6, 7$ will be related to *electro-weak* interactions and the vector potential components:

$$a_\mu^{(\underline{s})}, \qquad \underline{s} = 4, 5, 6, 7, \qquad \text{(electro-weak interaction field)} \qquad (6.76)$$

will be interpreted as *electro-weak fields.* The space-time dimensionality required, is now raised up to $n = 8$. The *electromagnetic* and *weak* interaction fields are mixed in the unified *electro-weak* theory. So the choice of the physical meaning of these vector potentials $a_\mu^{(\underline{s})}$ will depend on the *standard model* representation adopted.

Non-Diagonal Representation

The non-diagonal representation of the *electro-weak* field involves the vector fields $B_\mu, W_\mu^a, a = \mathbf{1}, \mathbf{2}, \mathbf{3},$

the corresponding strength tensor being given by:

$$F_{\mu\nu} = B_{\nu,\mu} - B_{\mu,\nu},$$
$$F^a_{\mu\nu} = W^a_{\nu,,\mu} - W^a_{\mu,,\nu} + g\epsilon^a{}_{bc} W^b_\mu W^c_\nu, \tag{6.77}$$

where g is one of the *electro-weak* coupling constants and $\epsilon^a{}_{bc}$ is the Levi-Civita symbol. So we are led to associate the components of each potential $a^{(\underline{s})}_\mu$ in physical space-time V^4 ($\mu = 0, 1, 2, 3$) in the following way:

$$a^{(4)}_\mu = B_\mu, \quad a^{(5)}_\mu = W^{\mathbf{1}}_\mu, \quad a^{(6)}_\mu = W^{\mathbf{2}}_\mu, \quad a^{(7)}_\mu = W^{\mathbf{3}}_\mu. \tag{6.78}$$

Dimensional constants depending on the unit system adopted have been dropped here to avoid heavier notations, being absorbed into the definition of the fields themselves. Then the strength field tensors components result:

$$f^{(\underline{s})}_{\mu\nu} = a^{(\underline{s})}_{\nu,\mu} - a^{(\underline{s})}_{\mu,\nu} + C^{(\underline{s})}{}_{(\underline{r})(\underline{q})(0)}\, a^{(\underline{r})}_\mu a^{(\underline{q})}_\nu. \tag{6.79}$$

Identifying (6.77) and (6.79) we determine the structure constants:

$$C^{(4)}{}_{(\underline{r})(\underline{q})(0)} = 0, \qquad C^{(\underline{s})}{}_{(\underline{r})(\underline{q})(0)} = \delta^{(\underline{q}-3)}_a \delta^b_{(\underline{r}-3)} \delta^c_{(\underline{q}-3)}\, g\epsilon^a{}_{bc}. \tag{6.80}$$

Diagonal Representation

According to the *standard model* the physical fields:

$$a^{(\underline{4})}_\mu = A_\mu, \quad a^{(\underline{5})}_\mu = W^{\mathbf{1}}_\mu, \quad a^{(\underline{6})}_\mu = W^{\mathbf{2}}_\mu, \quad a^{(\underline{7})}_\mu = Z_\mu, \tag{6.81}$$

are provided by the *diagonal representation*, which is obtained thanks to a rotation of $W^{\mathbf{3}}_\mu, B_\mu$ of the Weinberg angle, defined by the relation:

$$\tan\theta_W = \frac{g'}{g}, \tag{6.82}$$

g' being a second *electro-weak* coupling constant. Then we have the following alternative way to associate our vector potentials to the *electro-weak* fields.

$$\begin{aligned} Z_\mu &= W_\mu^3 \cos\theta_W - B_\mu \sin\theta_W, \\ A_\mu &= W_\mu^3 \sin\theta_W + B_\mu \cos\theta_W. \end{aligned} \tag{6.83}$$

In the diagonal representations the strength tensors are given by substitution of the inverse rotation:

$$\begin{aligned} W_\mu^3 &= Z_\mu \cos\theta_W + A_\mu \sin\theta_W, \\ B_\mu &= -Z_\mu \sin\theta_W + A_\mu \cos\theta_W. \end{aligned} \tag{6.84}$$

into (6.77). We have:

$$\begin{aligned} F_{\mu\nu} &= -F^Z_{\mu,\nu} \sin\theta_W + F^A_{\mu,\nu} \cos\theta_W, \\ F^{\mathbf{3}}_{\mu\nu} &= F^Z_{\mu,\nu} \cos\theta_W + F^A_{\mu,\nu} \sin\theta_W + g\epsilon^{\mathbf{3}}{}_{bc} W^b_\mu W^c_\nu, \end{aligned} \tag{6.85}$$

with:

$$F^Z_{\mu\nu} = Z_{\nu,\mu} - Z_{\mu,\nu}, \qquad F^A_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}. \tag{6.86}$$

Finally the strength tensor components in the diagonal representations results:

$$\begin{aligned} F^A_{\mu\nu} &= A_{\nu,\mu} - A_{\mu,\nu}, \\ F^{\mathbf{1}}_{\mu\nu} &= W^{\mathbf{1}}_{\nu,,\mu} - W^{\mathbf{1}}_{\mu,,\nu} + g\epsilon^{\mathbf{1}}{}_{bc} W^b_\mu W^c_\nu, \\ F^{\mathbf{2}}_{\mu\nu} &= W^{\mathbf{2}}_{\nu,,\mu} - W^{\mathbf{2}}_{\mu,,\nu} + g\epsilon^{\mathbf{2}}{}_{bc} W^b_\mu W^c_\nu, \\ F^Z_{\mu\nu} &= Z_{\nu,\mu} - Z_{\mu,\nu}. \end{aligned} \tag{6.87}$$

Identification of (6.87) with (6.79) leads to:

$$f^{(\underline{4})}_{\mu\nu} = F^{A}_{\mu\nu}, \quad f^{(\underline{5})}_{\mu\nu} = F^{\mathbf{1}}_{\mu\nu}, \quad f^{(\underline{6})}_{\mu\nu} = F^{\mathbf{2}}_{\mu\nu}, \quad f^{(\underline{7})}_{\mu\nu} = F^{Z}_{\mu\nu}, \tag{6.88}$$

and determines the relations for the structure constants:

$$\begin{aligned}
C^{(\underline{4})}{}_{(\underline{r})(\underline{q})(0)} &= 0, \\
C^{(\underline{5})}{}_{(\underline{r})(\underline{q})(0)} &= \delta^{b}_{(\underline{r}-3)}\delta^{c}_{(\underline{q}-3)}\, g\,\epsilon^{\mathbf{1}}{}_{bc}, \\
C^{(\underline{6})}{}_{(\underline{r})(\underline{q})(0)} &= \delta^{b}_{(\underline{r}-3)}\delta^{c}_{(\underline{q}-3)}\, g\,\epsilon^{\mathbf{2}}{}_{bc}, \\
C^{(\underline{7})}{}_{(\underline{r})(\underline{q})(0)} &= 0.
\end{aligned} \tag{6.89}$$

In principle *any* other representation is allowed, each one identifying the correspondent structure constants. In each representation the field equations exhibit the Maxwellian form:

$$\mathrm{D}_{\alpha} f^{\alpha}_{(\underline{s})\mu} = J_{(\underline{s})\mu}, \qquad \underline{s} = 4, 5, 6, 7. \tag{6.90}$$

We will examine the current density $J_{(\underline{s})\mu}$ in the next chapter, dealing with fermions. When expressed in terms of the vector potentials the previous equation becomes, in the Lorentz gauge, a Klein-Gordon equation with a current density.

6.3.3 The Strong Interaction Field

Strong interaction field carried by massless *gluons* remain to be introduced into the theory. We need other 8 non-Abelian fields $A^{\mathcal{A}}_{\mu}$ which we associate to the indices $\underline{s} = 8, 9, \cdots, 15$:

$$a^{(\underline{s})}_{\mu} = \delta^{(\underline{s}-7)}_{\mathcal{A}} A^{\mathcal{A}}_{\mu}, \qquad \underline{s} = 8, 9, \cdots, 15 \quad \mathcal{A} = 1, 2, \cdots, 8. \tag{6.91}$$

We conclude that $n = 16$ space-time dimensions are required to describe all the known fundamental interactions. The strength tensors are now:

$$F^{\mathcal{A}}_{\mu\nu} = A^{\mathcal{A}}_{\nu,\mu} - A^{\mathcal{A}}_{\mu,\nu} + g_s \mathcal{C}^{\mathcal{A}}{}_{\mathcal{B}\mathcal{C}} A^{\mathcal{B}}_{\mu} A^{\mathcal{C}}_{\nu}. \tag{6.92}$$

where g_s is a coupling constant for strong interactions and the structure constants, according to the *standard model,* are given by (see, [20]):

$$\begin{aligned} &\mathcal{C}_{123} = 1, \quad \mathcal{C}_{147} = \tfrac{1}{2}, \quad \mathcal{C}_{156} = -\tfrac{1}{2}, \quad \mathcal{C}_{246} = \tfrac{1}{2}, \quad \mathcal{C}_{257} = \tfrac{1}{2}, \\ &\mathcal{C}_{345} = \tfrac{1}{2}, \quad \mathcal{C}_{367} = -\tfrac{1}{2}, \quad \mathcal{C}_{458} = \sqrt{\tfrac{3}{2}}, \quad \mathcal{C}_{678} = \sqrt{\tfrac{3}{2}}. \end{aligned} \tag{6.93}$$

Identification of (6.79) with (6.92) now yields:

$$C^{(\underline{s})}{}_{(\underline{r})(\underline{q})} = \delta^{(\underline{s}-7)}_{\mathcal{A}} \delta^{\mathcal{B}}_{(\underline{r}-7)} \delta^{\mathcal{C}}_{(\underline{q}-7)} g_s \mathcal{C}^{\mathcal{A}}{}_{\mathcal{B}\mathcal{C}}. \tag{6.94}$$

The Maxwellian field equations are now:

$$\mathrm{D}_{\alpha} f^{\alpha}_{(\underline{s})\mu} = J_{(\underline{s})\mu}, \qquad \underline{s} = 8, 9, \cdots, 15. \tag{6.95}$$

Also for strong interactions the current density term will be examined in the next chapter.

All the previous results concerning the interactions fields can be summarized in the following scheme.

Unified interaction field equations

(*gravitational*) $$R^{<4>}_{\mu\nu} - \frac{1}{2} R^{<4>} g_{\mu\nu} \overbrace{-\Lambda g_{\mu\nu} - \lambda^{[\underline{s}]} a_{(\underline{s})\mu} a^{(\underline{s})}_{\nu}}^{\textit{dark energy?}} = \kappa \Big(T^{[f]}_{\mu\nu} + \overbrace{T^{[d]}_{\mu\nu}}^{\textit{dark matter?}} \Big)$$

(*Abelian e-w*) $$f^{\alpha}_{(\underline{4})\,\mu|\alpha} = J_{(\underline{4})\mu},$$

(*non-Abelian e-w*) $$\mathrm{D}_\alpha f^{\alpha}_{(\underline{s})\,\mu} = J_{(\underline{s})\mu}, \quad \underline{s} = a + 4 = 5, 6, 7, \quad a = 1, 2, 3,$$

(*strong*) $$\mathrm{D}_\alpha f^{\alpha}_{(\underline{s})\,\mu} = J_{(\underline{s})\mu}, \quad \underline{s} = \mathcal{A} + 7 = 8, \cdots 15, \quad \mathcal{A} = 1, \cdots, 8.$$

In the next chapter we will be concerned with the *extra* components of the vector potentials $a^{(\bar{\sigma})}_{\underline{l}}, \underline{l} = 4, 5, \cdots, 15,$ and the respective field equations.

Chapter 7

Matter Fields (Fermions)

Abstract
The present chapter is devoted to show how the matter fields (*spinors*) required to describe the *leptons* and *quarks* appearing in the *standard model,* just fit the *extra* components of the vector potentials. Current densities are also examined.

7.1 Introduction

In the present chapter we are interested in interpreting the physical meaning of the 12 *extra* components of the field vector potentials, *i.e.,* $a_{\underline{l}}^{(\bar{\sigma})}, \underline{l} = 4, 5, \cdots, 15$.

We have just shown in chapter 6 as:

1. The first 4 components $a_{\mu}^{(\sigma)}$ of the 4 potentials $a_{\bar{\mu}}^{(\sigma)}, \sigma = 0, 1, 2, 3$, can be interpreted as related to the *gravitational field* in the physical observable space-time V^4;
2. The first 4 components $a_{\mu}^{(4)}$ of the potential $a_{\bar{\mu}}^{(4)}$ can be interpreted as related to an *Abelian* (*electromagnetic*) counterpart of the *electro-weak field* in V^4;
3. The first 4 components $a_{\mu}^{(4+a)}, a = \mathbf{1}, \mathbf{2}, \mathbf{3}$ of the 3 potentials $a_{\bar{\mu}}^{(4+a)}$ can be interpreted as related to the *non-Abelian* (*weak*) counterpart of the *electro-weak field* in V^4;
4. The first 4 components $a_{\mu}^{(7+\mathcal{A})}, \mathcal{A} = 1, 2, \cdots, 8$ of the 8 potentials $a_{\bar{\mu}}^{(7+\mathcal{A})}$ can be interpreted as related to the *strong interaction field* in V^4.

In particular we precise here that, according to the *standard model*, for "*Abelian* counterpart of the *electro-weak field*" we mean

- either the Abelian field B_μ (in non-diagonal representation),
- or respectively the electromagnetic field A_μ (in diagonal representation).

And for "*non-Abelian* counterpart of the *electro-weak field*" we intend

- either the non-Abelian field W_μ^a of the unified electro-weak theory (in non-diagonal representation),
- or respectively the correspondent components $Z, W_\mu^{\mathbf{1}}, W_\mu^{\mathbf{2}}$ of the weak field, rotated by the Weinberg angle (in diagonal representation).

The residual 12 *extra* components $a_{\underline{l}}^{(\bar{\sigma})}, \underline{l} = 4, 5, \cdots, 15$, of each potential associated to the physical interactions (*i.e., gravitational, electro-weak* and *strong*), can be seen by an observer living within the physical space-time V^4, as fields which behave as *scalars,* respect to the group of Riemannian 4-*dimensional* co-ordinate transformations.

The problem of interpreting as physically meaningful those $12 \times 16 = 192$ scalar fields is now to be attacked. We emphasize that the latter exceeding components behave as scalars respect to curvilinear co-ordinate transformation in physical space-time V^4, and fulfil D'Alembert or Klein-Gordon equations (with current densities).

Then, in principle, we could guess that, at least some of them might be combined into Dirac spinors which satisfy Dirac equations (and then also Kein-Gordon equations).

In particular, according to the elementary particle *standard model,* we need 12 Dirac spinors: 6 for the *leptons* $e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau$, and 6 for the quarks *up, down, top, bottom, charm, strange,* respectively with left and right chirality and 12 more ones for the correspondent *anti*-particles. Since each spinor has 4 components then $12 \times 2 \times 4 = 96$ are required for particles and 96 for *anti*-particles *i.e.,* 192 functions.

7.2 Field Extra Components

Let us, now, examine in some detail the components $a_{\bar{\mu}}^{(\bar{\sigma})}$ of each vector potential appearing in the theory.

In the previous chapters we proposed to interpret the first 4 components of each potential as related to the fundamental interactions (*gravitational, electro-weak and strong*).

But in the 16-*dimensional* extended space-time V^{16}, each vector potential $\boldsymbol{a}^{(\bar{\sigma})} \equiv (a_{\bar{\mu}}^{(\bar{\sigma})})$, $\bar{\mu}, \bar{\sigma} = 0, 1, \cdots, 15$, exhibits, beside the 4 components labelled by the index $\mu = 0, 1, 2, 3$, within the physical space-time V^4, also 12 *extra* components labelled by the index $\underline{l} = 4, 5, \cdots, 15$.

As we have shown, the latter ones are seen as scalar fields by an observer measuring them in the physical space-time V^4, since they are not affected by the transformations of the co-ordinates x^0, x^1, x^2, x^3 in V^4.

Until now we considered only interaction fields, carried by *bosons* (*gravitons, photons,* $W^{\pm}, Z^0$ and *gluons*) and nothing was said about *fermions* (*leptons* and *quarks*).

In order to complete the theory, now we introduce also the 6 *leptons* $(e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau)$ and the 6 *quarks,* (*up, down, top, bottom, charm, strange*) with their respective *anti*-particles and chirality.

So we may sketch the physical interpretation of the potential components according to the following scheme.

$$
\begin{array}{r c l}
 & \begin{array}{cc} \textit{bosons} & \textit{fermions} \\ \downarrow & \downarrow \end{array} & \\
\begin{array}{r} \textit{gravitational} \rightarrow \\ \\ \textit{Abelian electro-weak} \rightarrow \\ \\ \textit{non-Abelian electro-weak} \rightarrow \\ \\ \textit{strong} \rightarrow \end{array} &
\left(\begin{array}{ccc} (a_\mu^{(\sigma)} & , & a_{\underline{l}}^{(\sigma)}) \\ \\ (a_\mu & , & a_{\underline{l}}) \\ \\ (a_\mu^a & , & a_{\underline{l}}^a) \\ \\ (a_\mu^{\mathcal{A}} & , & a_{\underline{l}}^{\mathcal{A}}) \end{array}\right) &
\begin{array}{l} \leftarrow \textit{spinor components} \\ \\ \equiv \left(a_{\bar{\mu}}^{(\bar{\sigma})} \right) \\ \\ \leftarrow \textit{spinor components} \end{array} \\
 & \begin{array}{cc} \uparrow & \uparrow \\ V^4\textit{-vectors} & V^4\textit{-scalars (leptons, quarks)} \end{array} &
\end{array}
$$

7.3 Physical Meaning of the Field *Extra* Components

In this section we want to explain in more detail how the 192 *extra* components of the vector potentials $a_{\bar{\mu}}^{(\bar{\sigma})}$, may be related to the spinor fields associated to the *fermions* (*leptons* and *quarks*) appearing in the *standard model* of elementary particle theory.

Each spinor is a set of 4 complex valued functions of the observable co-ordinates x^μ, which are to be provided by the 192 complex functions offered by the *extra* components of the vector potentials $a_{\underline{l}}^{(\bar{\sigma})}$:

$$
\psi \equiv \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{pmatrix} . \tag{7.1}
$$

Every component of ψ can be thought as a linear combination of the components $a_{\underline{l}}^{(\bar{\sigma})}$:

$$\psi_{\underline{l}}^{(\bar{\sigma})} = \alpha_{\underline{l}}{}^{\underline{r}}\, a_{\underline{r}}^{(\bar{\sigma})}, \tag{7.2}$$

where $\alpha_{\underline{l}}{}^{\underline{r}}$ are the elements of a constant matrix, the choice of which leads to one of the possible representations of the same fermions.

The simplest representation is given, of course, by:

$$\psi_{\underline{l}}^{(\bar{\sigma})} = a_{\underline{l}}^{(\bar{\sigma})}, \qquad \underline{l} = 4, 5, \cdots, 15, \tag{7.3}$$

which can always be obtained with a suitable choice of the *extra* co-ordinates $x^{\underline{i}}$.

Then we can associate groups of 4 components to the spinors representing the physical elementary fermions, *e.g.,* as:

$$\begin{pmatrix} \psi_{\underline{l}}^{(\bar{\sigma})} \\ \psi_{\underline{l}}^{(\bar{\sigma}+1)} \\ \psi_{\underline{l}}^{(\bar{\sigma}+2)} \\ \psi_{\underline{l}}^{(\bar{\sigma}+3)} \end{pmatrix}, \quad \begin{aligned} &\underline{l} = 4, 5, \cdots, 15, \\ &\bar{\sigma} = 0 \ \ (\textit{l.h. and r.h. leptons}), \\ &\bar{\sigma} = 4 \ \ (\textit{l.h. and r.h. red quarks}), \\ &\bar{\sigma} = 8 \ \ (\textit{l.h. and r.h. green quarks}), \\ &\bar{\sigma} = 12 \ \ (\textit{l.h. and r.h. blue quarks}), \end{aligned} \tag{7.4}$$

where *l.h., r.h.* denote respectively left-hand and right-hand *chirality* and *red, green, blue* quark *color*.

According to this representation a detailed scheme of the physical meaning of all the components of the vector potentials $a_{\underline{\mu}}^{(\bar{\sigma})}$ is summarized in the following scheme.

$$
(a_{\bar\mu}^{(\bar\sigma)}) \equiv \left(\begin{array}{c|cccc|cccccc|c}
 & \multicolumn{4}{c|}{\textit{bosons}} & \multicolumn{6}{c|}{\textit{fermions}} & \\
 & & & & & \multicolumn{6}{c|}{e_L\, e_R\, \mu_L\, \mu_R\, \tau_L\, \tau_R\, \nu_{eL}\, \nu_{eR}\, \nu_{\mu L}\, \nu_{\mu R}\, \nu_{\tau L}\, \nu_{\tau R}} & \\
\textit{gravitons} & a_0^{(0)} & a_1^{(0)} & a_2^{(0)} & a_3^{(0)} & a_4^{(0)} & \cdots & a_9^{(0)} & a_{10}^{(0)} & \cdots & a_{15}^{(0)} & \textit{leptons} \\
 & a_0^{(1)} & a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_4^{(1)} & \cdots & a_9^{(1)} & a_{10}^{(1)} & \cdots & a_{15}^{(1)} & \\
 & a_0^{(2)} & a_1^{(2)} & a_2^{(2)} & a_3^{(2)} & a_4^{(2)} & \cdots & a_9^{(2)} & a_{10}^{(2)} & \cdots & a_{15}^{(2)} & \\
 & a_0^{(3)} & a_1^{(3)} & a_2^{(3)} & a_3^{(3)} & a_4^{(3)} & \cdots & a_9^{(3)} & a_{10}^{(3)} & \cdots & a_{15}^{(3)} & \\
\gamma, W^{\pm}, Z^0 & a_0^{(4)} & a_1^{(4)} & a_2^{(4)} & a_3^{(4)} & a_4^{(4)} & \cdots & a_9^{(4)} & a_{10}^{(4)} & \cdots & a_{15}^{(4)} & \textit{blue quarks} \\
 & a_0^{(5)} & a_1^{(5)} & a_2^{(5)} & a_3^{(5)} & a_4^{(5)} & \cdots & a_9^{(5)} & a_{10}^{(5)} & \cdots & a_{15}^{(5)} & \\
 & a_0^{(6)} & a_1^{(6)} & a_2^{(6)} & a_3^{(6)} & a_4^{(6)} & \cdots & a_9^{(6)} & a_{10}^{(6)} & \cdots & a_{15}^{(6)} & \\
 & a_0^{(7)} & a_1^{(7)} & a_2^{(7)} & a_3^{(7)} & a_4^{(7)} & \cdots & a_9^{(7)} & a_{10}^{(7)} & \cdots & a_{15}^{(7)} & \\
\textit{gluons} & a_0^{(8)} & a_1^{(8)} & a_2^{(8)} & a_3^{(8)} & a_4^{(8)} & \cdots & a_9^{(8)} & a_{10}^{(8)} & \cdots & a_{15}^{(8)} & \textit{green quarks} \\
 & a_0^{(9)} & a_1^{(9)} & a_2^{(9)} & a_3^{(9)} & a_4^{(9)} & \cdots & a_9^{(9)} & a_{10}^{(9)} & \cdots & a_{15}^{(9)} & \\
 & a_0^{(10)} & a_1^{(10)} & a_2^{(10)} & a_3^{(10)} & a_4^{(10)} & \cdots & a_9^{(10)} & a_{10}^{(10)} & \cdots & a_{15}^{(10)} & \\
 & a_0^{(11)} & a_1^{(11)} & a_2^{(11)} & a_3^{(11)} & a_4^{(11)} & \cdots & a_9^{(11)} & a_{10}^{(11)} & \cdots & a_{15}^{(11)} & \\
 & a_0^{(12)} & a_1^{(12)} & a_2^{(12)} & a_3^{(12)} & a_4^{(12)} & \cdots & a_9^{(12)} & a_{10}^{(12)} & \cdots & a_{15}^{(12)} & \textit{red quarks} \\
 & a_0^{(13)} & a_1^{(13)} & a_2^{(13)} & a_3^{(13)} & a_4^{(13)} & \cdots & a_9^{(13)} & a_{10}^{(13)} & \cdots & a_{15}^{(13)} & \\
 & a_0^{(14)} & a_1^{(14)} & a_2^{(14)} & a_3^{(14)} & a_4^{(14)} & \cdots & a_9^{(14)} & a_{10}^{(14)} & \cdots & a_{15}^{(14)} & \\
 & a_0^{(15)} & a_1^{(15)} & a_2^{(15)} & a_3^{(15)} & a_4^{(15)} & \cdots & a_9^{(15)} & a_{10}^{(15)} & \cdots & a_{15}^{(15)} & \\
 & & & & & \multicolumn{6}{c|}{up_L\, up_R\, dn_L\, dn_R\, tp_L\, tp_R\, bm_L\, b_R\, st_L\, st_R\, ch_L\, ch_R} & \\
 & \multicolumn{4}{c|}{\textit{bosons}} & \multicolumn{6}{c|}{\textit{fermions}} &
\end{array}\right)
$$

The index $\underline{l}$, running here from 4 to 15, labels 12 spinors corresponding to the 6 leptons $e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau$, and to the 6 quarks in dependence on the values of σ and 12 more spinors related to the respective *anti*-particles.

7.3.1 Dirac Equations Governing Matter Fields

The *extra* equations in the complement space of V^4 which govern fermion fields are given by [see eq (6.73) in §6.3 of chapter 6]:

$$R^{<4>}_{\mu\underline{l}} - \tfrac{1}{2} R^{<4>} g_{\mu\underline{l}} - \Lambda g_{\mu\underline{l}} - \lambda^{[\underline{s}]} a_{(\underline{s})\mu} a^{(\underline{s})}_{\underline{l}} = \kappa T_{\mu\underline{l}}, \tag{7.5}$$

$$R^{<4>}_{\underline{j}\underline{l}} - \tfrac{1}{2} R^{<4>} g_{\underline{j}\underline{l}} - \Lambda g_{\underline{j}\underline{l}} - \lambda^{[\underline{s}]} a_{(\underline{s})\underline{j}} a^{(\underline{s})}_{\underline{l}} = \kappa T_{\underline{j}\underline{l}}, \tag{7.6}$$

$$\mathrm{D}_{\bar\alpha}\, f^{\bar\alpha}_{(\bar\sigma)\underline{l}} = J_{(\bar\sigma)\underline{l}}. \tag{7.7}$$

The energy-momentum tensor includes the contribution of the interaction fields and that of the gravitational contribution arising from *extra* dimensions, *i.e.*:

$$T_{\bar{\mu}\bar{\nu}} = T^{[f]}_{\bar{\mu}\bar{\nu}} + T^{[d]}_{\bar{\mu}\bar{\nu}}, \tag{7.8}$$

$$\kappa T^{[f]}_{\bar{\mu}\bar{\nu}} = f^{(\underline{s})}_{\bar{\alpha}\bar{\mu}}\, f^{\bar{\alpha}}_{(\underline{s})\bar{\nu}} - \tfrac{1}{4} f^{(\underline{s})}_{\bar{\alpha}\bar{\beta}}\, f^{\bar{\alpha}\bar{\beta}}_{(\underline{s})}\, g_{\bar{\mu}\bar{\nu}}, \tag{7.9}$$

$$\kappa T^{[d]}_{\bar{\mu}\bar{\nu}} = R^{<4>}_{\bar{\mu}\bar{\nu}} - \overline{R}_{\bar{\mu}\bar{\nu}} - \tfrac{1}{2}\big(R^{<4>} - \overline{R}\big) g_{\bar{\mu}\bar{\nu}}. \tag{7.10}$$

According to the *standard model* the covariant derivatives are determined in such a way that the gauge invariance conditions in V^4 are preserved even when a gauge choice is fixed in the *extra* space-time. Such a choice is always possible because of the degrees of freedom provided by the *anti*-symmetric tensor $\mathcal{A}_{\bar{\mu}\bar{\nu}}$ (arbitrary until now). Moreover, thanks to the latter tensor we will be able to obtain also the correct current densities in the r.h.s. of the interaction fields equations.

Now if we consider (5.72), from chapter 5 with vanishing currents we have when $\bar{\mu} = \underline{l}$:

$$g^{\bar{\alpha}\bar{\beta}}\, \mathrm{D}_\alpha f^{(\bar{\sigma})}_{\bar{\beta}\underline{l}} = 0, \tag{7.11}$$

which, thanks to the Lorentz gauge, is equivalent to:

$$g^{\bar{\alpha}\bar{\beta}}\, \mathrm{D}_{\bar{\alpha}}\, \mathrm{D}_{\bar{\beta}}\, a^{(\bar{\sigma})}_{\underline{l}} = 0. \tag{7.12}$$

Following a scheme like (7.4) we may replace the second order equations (7.12) for the potentials $a^{(\bar{\sigma})}_{\underline{l}}$ with the second order spinor equations:

$$g^{\bar{\alpha}\bar{\beta}}\, \mathrm{D}_{\bar{\alpha}}\, \mathrm{D}_{\bar{\beta}}\, \psi^{(\bar{\sigma})}_{\underline{l}} = 0. \tag{7.13}$$

Rest masses and contributions are expected to be hidden into the derivatives respect to the *extra* co-ordinates, so that (7.13) identify with the Klein-Gordon equations:

$$g^{\alpha\beta}\, \mathrm{D}_\alpha \mathrm{D}_\beta\, \psi^{(\bar{\sigma})}_{\underline{l}} + \frac{m^2_{[\bar{\sigma}\,\underline{l}]}\, c^2}{\hbar^2}\, \psi^{(\bar{\sigma})}_{\underline{l}} = 0, \tag{7.14}$$

which lead to the Dirac equations:

$$\gamma^{\alpha} \mathrm{D}_{\alpha} \psi_{\underline{l}}^{(\bar{\sigma})} + i \, \frac{m_{[\bar{\sigma}\,\underline{l}]}\,c}{\hbar} \; \psi_{\underline{l}}^{(\bar{\sigma})} = 0, \tag{7.15}$$

$m_{[\bar{\sigma}\,\underline{l}]}$ being the respective rest masses of *leptons* and *quarks*, related to solutions arising from the *extra* components of the vector fields with masses:

$$m_{[\bar{\sigma}\,\underline{l}]} = g^{[\bar{\sigma}\,\underline{l}]} M. \tag{7.16}$$

7.3.2 Current Densities

Let us now examine the current densities $J_{\bar{\mu}}^{(\bar{\sigma})}$.

When $\bar{\mu} = \mu$, the corresponding 4-vector $J_{\mu}^{(\bar{\sigma})}$ is to be related to the physical charge current density:

$$j_{\bar{\mu}}^{(\bar{\sigma})} = e^{(\bar{\sigma})} \overline{\psi}_{[\bar{\sigma}]} \gamma_{\mu} \psi_{[\bar{\sigma}]} \delta_{\bar{\mu}}^{\mu} \qquad (\textit{no sum over } \bar{\sigma}), \tag{7.17}$$

where the notation $e^{(\bar{\sigma})}$ means each kind of charge carried by fermions.

Then this identification follows:

$$J_{\mu}^{(\bar{\sigma})} = e^{(\bar{\sigma})} \overline{\psi}_{[\bar{\sigma}]} \gamma_{\mu} \psi_{[\bar{\sigma}]}. \tag{7.18}$$

We remember that special care is required in managing the index notation since there is no sum over $\bar{\sigma}$ when it is enclosed into square brackets, while summation is intended when $\bar{\sigma}$ is enclosed into round parentheses.

Since $J_{\bar{\mu}}^{(\bar{\sigma})}$, from eq (5.65) with $n = 16$, results to be:

$$J_{\bar{\mu}}^{(\bar{\sigma})} = \mathcal{J}_{\bar{\mu}}^{(\bar{\sigma})} + \mathcal{A}_{\bar{\mu}}^{(\bar{\sigma})} + \tfrac{1}{32} f_{\bar{\alpha}\,\bar{\beta}}^{(\bar{\tau})} f_{(\bar{\tau})}^{\bar{\alpha}\,\bar{\beta}} a_{(\bar{\sigma})\bar{\mu}} + \lambda^{[\bar{\sigma}]} a_{\bar{\mu}}^{(\bar{\sigma})},$$

where [See chap 5, eqs (5.38) and (5.65)]:

$$\mathcal{J}_{\bar{\mu}}^{(\bar{\sigma})} = \mathcal{J}_{\bar{\mu}\,\bar{\nu}}\, a^{(\bar{\sigma})\bar{\nu}}, \qquad \mathcal{A}_{\bar{\mu}}^{(\bar{\sigma})} = \mathcal{A}_{\bar{\mu}\,\bar{\nu}}\, a^{(\bar{\sigma})\bar{\nu}}.$$

we can determine the until now free term $\mathcal{A}_{\bar{\mu}}^{(\bar{\sigma})}$ as:

$$\mathcal{A}_{\bar{\mu}}^{(\bar{\sigma})} = e^{(\bar{\sigma})}\overline{\psi}_{[\bar{\sigma}]}\boldsymbol{\gamma}_{\mu}\psi_{[\bar{\sigma}]}\delta_{\bar{\mu}}^{\mu} - \mathcal{J}_{\bar{\mu}}^{(\bar{\sigma})} - \tfrac{1}{32} f_{\bar{\alpha}\bar{\beta}}^{(\bar{\tau})} f_{(\bar{\tau})}^{\bar{\alpha}\bar{\beta}} a_{(\bar{\sigma})\bar{\mu}} - \lambda^{[\bar{\sigma}]} a_{\bar{\mu}}^{(\bar{\sigma})}. \qquad (7.19)$$

7.4 Conclusion

In the last two chapters we have proposed a possible physical interpretation of the model of unified interaction (*boson*) and matter (*fermion*) unified field within the geometry of a multidimensional space-time manifold V^{16}.

We have shown how to identify interaction fields with the vector components $a_{\mu}^{(\bar{\sigma})}, \mu = 0, 1, 2, 3$ of the eigenvectors $a_{\bar{\mu}}^{(\bar{\sigma})}, \bar{\mu} = 0, 1, 2, \cdots, 15$ of the metric tensor $\boldsymbol{g} \equiv (g_{\bar{\mu}\bar{\nu}})$ in V^{16}.

More we have seen how to identify *fermion* matter spinor fields with the *extra* components $a_{\underline{l}}^{(\bar{\sigma})}, \underline{l} = 4, 5, \cdots, 15$ of the same eigenvector potentials. The model seems to be suitable to agree with the *standard model* of elementary particles.

In order to complete a unified description of universe it remains to examine if and how the proposed theory can be suitable to provide meaningful results also in cosmology and if it may open a way to quantization of the gravitational field.

The last two chapters of the book will be just involved with such intriguing topics.

Chapter 8

Cosmology and Elementary Particles

Abstract
This chapter is intended to provide a possibile application to cosmology of the theory presented in the previous chapters. We test a reasonably simple diagonal metric solution of the Einstein equations in empty extended space-time in presence of the cosmological constant and interpret the energy-momentum contributions appearing in the observable 4-*dimensional* space-time as related to the interaction and matter fields and the residual contribution as related to *dark matter* and *dark energy.* It is remarkable that the flatness of the universe appears naturally for the examined solution. Geodesic motion is also analyzed.

8.1 Introduction

This chapter is devoted to cosmological applications of the theory proposed in the previous parts of the book. Some remarkable results will arise thanks to the extended dimensionality of space-time.

1. First of all the *flatness* of universe will arise from compatibility of the *space-space* components of the Einstein cosmological equations for a simple diagonal metric solution extending the Robertson-Walker metric to V^{16}.

2. As a second result, *dark energy* and *dark matter* contributions will appear as owed to gravity hidden within the *extra* dimensions of space-time.

3. Moreover the introduction of the vector potentials $a_{\bar{\mu}}^{(\bar{\sigma})}$, in the representation of the metric tensor, will allow to build up for each field, Hamiltonian densities the structure of which is similar to the well known Hamiltonian of the electromagnetic field. Then the Hamiltonian of each kind of field may be quantized in an usual way, like in *q.e.d.*.

   Therefore also a way to quantization of the gravitational field appears be open (see the next chapter 9).

8.2 Extension of Robertson-Walker Metric in V^{16}

Let us now introduce the cosmological metric in the physical sub-space-time V^4 embedded within the extended V^{16} space-time. The co-ordinates are given by:

$$x^0 = ct, \qquad x^1 = r, \quad x^2 = \theta, \quad x^3 = \phi, \\ x^{\underline{i}}, \quad \underline{i} = 4, 5, \cdots, 15. \tag{8.1}$$

In order to preserve the cosmological principle in V^4 the metric tensor components $g_{\mu\nu}$ are required to be still the usual Robertson-Walker metric components:

$$g_{00} = 1, \qquad g_{11} = -\frac{a(t)^2}{1 - Kr^2}, \\ g_{22} = -a(t)^2 r^2, \qquad g_{33} = -a(t)^2 r^2 \sin^2\theta. \tag{8.2}$$

Additional components $g_{\mu\underline{l}}, g_{\underline{il}}$ required to extend the metric to the entire V^{16} will be added in the *extra-space.* Then the metric tensor representation is given by:

$$\boldsymbol{g} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & g_{0\underline{l}} \\ 0 & -\frac{a(t)^2}{1-Kr^2} & 0 & 0 & g_{1\underline{l}} \\ 0 & 0 & -a(t)^2 r^2 & 0 & g_{2\underline{l}} \\ 0 & 0 & 0 & -a(t)^2 r^2 \sin^2\theta & g_{3\underline{l}} \\ g_{\underline{i}0} & g_{\underline{i}1} & g_{\underline{i}2} & g_{\underline{i}3} & g_{\underline{il}} \end{pmatrix}. \tag{8.3}$$

8.3 The Cosmological Field Equations in V^{16}

We assume that the cosmological field equations, when written in the extended space-time V^{16}, are given by the Einstein equations in empty space-time, so that any fundamental field living in the observable physical space-time V^4 results to be included into the V^{16} metric tensor $g_{\bar{\mu}\bar{\nu}}$.

We have:

$$R_{\bar{\mu}\bar{\nu}} - \tfrac{1}{2} R g_{\bar{\mu}\bar{\nu}} - \Lambda g_{\bar{\mu}\bar{\nu}} = 0. \tag{8.4}$$

It is remarkable that the equations (8.4), being written in *empty* V^{16} space-time are not affected if the metric tensor is multiplied by an arbitrary non-vanishing factor, except for the same factor multiplying the cosmological constant which results simply to be rescaled.

In general the latter equations are very hard to solve, unless the metric is diagonal within the whole space-time V^{16}.

8.3.1 Diagonal Solution and Universe Flatness

A simple and meaningful solution is immediately obtained if we assume that the metric is diagonal within the whole extended space-time V^{16}, and exhibits the form:

$$g_{\mu\underline{l}} = 0, \qquad g_{\underline{i}\underline{l}} = -|c_{[\underline{l}]}|^2 a(t)^2 \delta_{\underline{i}\underline{l}}, \tag{8.5}$$

$c_{[\underline{l}]}$ being a constant to be determined and physically interpreted later. Conveniently we redefine also the scale of the ordinary space-time components $g_{\mu\nu}$ as follows:

$$\begin{gathered} g_{00} = |c_g|^2, \qquad g_{11} = -\frac{|c_g|^2 a(t)^2}{1 - Kr^2}, \\ g_{22} = -|c_g|^2 a(t)^2 r^2, \qquad g_{33} = -|c_g|^2 a(t)^2 \sin^2\theta, \end{gathered} \tag{8.6}$$

so that a constant coefficient $|c_g|^2$ may appear also in $g_{00}, g_{11}, g_{22}, g_{33}$. As we will see in chapter 9 the coefficients $c_g, c_{[\underline{l}]}$ will play an important role in order to the field quantization.

Therefore the matrix representation of the metric tensor and its inverse result to be:

$$g \equiv \begin{pmatrix} |c_g|^2 & 0 & 0 & 0 & 0 \\ 0 & -\frac{|c_g|^2 a(t)^2}{1-Kr^2} & 0 & 0 & 0 \\ 0 & 0 & -|c_g|^2 a(t)^2 r^2 & 0 & 0 \\ 0 & 0 & 0 & -|c_g|^2 a(t)^2 r^2 \sin^2\theta & 0 \\ 0 & 0 & 0 & 0 & -|c_{[\underline{l}]}|^2 a(t)^2 \delta_{\underline{il}} \end{pmatrix}, \quad (8.7)$$

$$g^{-1} \equiv \begin{pmatrix} \frac{1}{|c_g|^2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1-Kr^2}{|c_g|^2 a(t)^2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{|c_g|^2 a(t)^2 r^2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{|c_g|^2 a(t)^2 r^2 \sin^2\theta} & 0 \\ 0 & 0 & 0 & 0 & -\frac{\delta_{\underline{il}}}{|c_{[\underline{l}]}|^2 a(t)^2} \end{pmatrix}. \quad (8.8)$$

In correspondence to this solution the non-vanishing components of the connection coefficients become:

$$\begin{aligned}
&\Gamma^{\bar{i}}_{0\bar{i}} = \frac{\dot{a}(t)}{ca(t)}, && \bar{i} = 1, 3, \cdots, 15, && \\
&\Gamma^0_{11} = \frac{a(t)\,\dot{a}(t)}{c(1-Kr^2)}, && \Gamma^1_{11} = \frac{Kr}{1-Kr^2}, && \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r}, \\
&\Gamma^0_{22} = \frac{r^2 a(t)\,\dot{a}(t)}{c}, && \Gamma^1_{22} = -r(1-Kr^2), && \Gamma^3_{23} = \cot\theta, \quad (8.9) \\
&\Gamma^0_{33} = \frac{r^2 a(t)\,\dot{a}(t)\sin^2\theta}{c}, && && \Gamma^1_{33} = -r(1-Kr^2)\,sin^2\theta, \\
&\Gamma^2_{33} = -\cos\theta\sin\theta, && \Gamma^0_{\underline{jj}} = \frac{|c_{[\underline{l}]}|^2 a(t)\,\dot{a}(t)}{|c_g|^2 c}, && \underline{j}, = 4, 5, \cdots, 15,
\end{aligned}$$

where no sum is intended over repeated index $\underline{j}$ in the last term.

The non vanishing Ricci tensor components are now given by:

$$R_{00} = -\frac{15\ddot{a}(t)}{c^2 a(t)}, \quad (8.10)$$

$$R_{11} = \frac{a(t)\ddot{a}(t) + 14\dot{a}(t)^2 + 2Kc^2}{c^2(1-Kr^2)}, \quad (8.11)$$

$$R_{22} = \frac{r^2\big[a(t)\ddot{a}(t) + 14\dot{a}(t)^2 + 2Kc^2\big]}{c^2}, \tag{8.12}$$

$$R_{33} = \frac{r^2\big[a(t)\ddot{a}(t) + 14\dot{a}(t)^2 + 2Kc^2\big]\sin^2\theta}{c^2}, \tag{8.13}$$

$$R_{\underline{j}\underline{j}} = \frac{|c_{[\underline{j}]}|^2\big[a(t)\ddot{a}(t) + 14\dot{a}(t)^2\big]}{|c_g|^2c^2} \qquad \text{(non sum over } \underline{j}\text{)}, \tag{8.14}$$

the Ricci scalar curvature resulting:

$$R = -\frac{6\big[5a(t)\ddot{a}(t) + 35\dot{a}(t)^2 + Kc^2\big]}{|c_g|^2c^2}. \tag{8.15}$$

Then the only non-trivial Einstein equations in empty V^{16} space-time become:

$$105\dot{a}(t)^2 - |c_g|^2\Lambda c^2 a(t)^2 + 3Kc^2 = 0, \tag{8.16}$$

in correspondence to the *time-time* component in (8.4). And:

$$14a(t)\ddot{a}(t) + 91\dot{a}^2(t) - |c_g|^2\Lambda c^2 a(t)^2 + Kc^2 = 0, \tag{8.17}$$

in correspondence to the observable *space-space* components. Moreover we have for the *extra space-space* components:

$$14a(t)\ddot{a}(t) + 91\dot{a}^2(t) - |c_g|^2\Lambda c^2 a(t)^2 + 3Kc^2 = 0. \tag{8.18}$$

Then the first new remarkable result appears immediately, thanks to the presence of *extra* dimensions, since compatibility between (8.17) and (8.18) requires:

$$K = 0, \tag{8.19}$$

so providing a possible explanation of the observed *flatness* of the physical universe, at least in correspondence to the examined diagonal solution. Then only the following equations remain:

$$105\dot{a}(t)^2 - |c_g|^2\Lambda c^2 a(t)^2 = 0, \tag{8.20}$$

$$14a(t)\ddot{a}(t) + 91\dot{a}^2(t) - |c_g|^2 \Lambda c^2 a(t)^2 = 0, \tag{8.21}$$

in which the coefficients $|c_{[l]}|^2$ do not appear, while the coefficient $|c_g|^2$, related to the gravitational field in V^4, does. Then from (8.20) one obtains, for positive Λ (as it is physically observed):

$$\frac{\dot{a}(t)}{ca(t)} = \pm|c_g| \sqrt{\tfrac{\Lambda}{105}}, \tag{8.22}$$

the integration of which leads to:

$$a(t) = a_0^{[\pm]} e^{\pm|c_g|\sqrt{\frac{\Lambda}{105}}\,ct}, \tag{8.23}$$

since the empty extended space-time V^{16} behaves like a multidimensional De Sitter universe, in which no singularity appears. Substitution of (8.23) into (8.21) provides:

$$|c_g|^2 c^2 a_0^{(\pm)} \Lambda e^{\pm 2|c_g|\sqrt{\frac{\Lambda}{105}}\,ct}\left(\tfrac{14}{105} + \tfrac{91}{105} - 1\right) \equiv 0,$$

so fulfilling also eq (8.21). Positive sign in the exponential corresponds to an expanding universe as it is physically observed.

8.3.2 What is Observed in the Physical Space-Time V^4

In the physical space-time V^4 the ordinary Robertson-Walker metric tensor $g_{\mu\nu}^{<4>}$, when $K = 0$, leads to the non-vanishing Ricci tensor components:

$$R_{00}^{<4>} = -\frac{3\ddot{a}(t)}{c^2 a(t)}, \tag{8.24}$$

$$R_{11}^{<4>} = \frac{a(t)\ddot{a}(t) + 2\dot{a}(t)^2}{c^2}, \tag{8.25}$$

$$R_{22}^{<4>} = \frac{r^2\left[a(t)\ddot{a}(t) + 2\dot{a}(t)^2\right]}{c^2}, \tag{8.26}$$

$$R_{33}^{<4>} = \frac{\left[a(t)\ddot{a}(t) + 2\dot{a}(t)^2\right]\sin^2\theta}{c^2}, \tag{8.27}$$

and the Ricci scalar curvature:

$$R^{<4>} = -\frac{6[a(t)\ddot{a}(t) + \dot{a}(t)^2]}{|c_g|^2 c^2 a(t)^2}. \tag{8.28}$$

Then the non-vanishing Einstein field equations in V^4 in presence of external matter-energy fields result:

$$3\frac{\dot{a}(t)^2}{c^2 a(t)^2} - |c_g|^2 \Lambda = \kappa \varrho c^2 |c_g|^2, \tag{8.29}$$

$$-2\frac{\ddot{a}(t)}{c^2 a(t)} - \frac{\dot{a}^2(t)}{c^2 a(t)^2} + |c_g|^2 \Lambda = \kappa \wp |c_g|^2, \tag{8.30}$$

where matter-energy fields are represented, as usual, as a perfect fluid of energy-momentum tensor:

$$T_{\mu\nu} = \left(\varrho c^2 + \wp\right) u_\mu u_\nu - \wp g_{\mu\nu}, \tag{8.31}$$

u^μ being the 4-velocity of the fluid particle, which in a co-moving reference, where:

$$u_0 = \sqrt{g_{00}} \equiv |c_g|, \qquad u_k = 0, \tag{8.32}$$

assumes the form:

$$T_{00} = \varrho c^2 |c_g|^2, \qquad T_{jk} = -\wp g_{jk}, \tag{8.33}$$

ϱ, $\wp$ being the mass-energy and pressure densities of the fluid.

Some care is required, since all the wave-particles travel at the speed of light in V^{16}, so that $u^{\bar{\mu}} u_{\bar{\mu}} \equiv u^\mu u_\mu + u^{\underline{i}} u_{\underline{i}} = u^0 u_0 + u^i u_i + u^{\underline{i}} u_{\underline{i}} = 0$; then thanks to (8.32-b) it follows $u^{\underline{i}} u_{\underline{i}} = -1$. So when the wave-particles travel at speed c in V^4 a co-moving frame can be defined only as a limiting case.

From (8.20) and (8.21) we solve:

$$\frac{\dot{a}(t)^2}{c^2 a(t)^2} = \tfrac{1}{105} \Lambda |c_g|^2, \tag{8.34}$$

$$\frac{\ddot{a}(t)}{c^2 a(t)} = \tfrac{1}{105} \Lambda |c_g|^2, \tag{8.35}$$

which substituted into (8.29) and (8.30) lead to:

$$\kappa \varrho c^2 = -\tfrac{34}{35} \Lambda, \tag{8.36}$$

$$\kappa \wp = \tfrac{34}{35} \Lambda, \tag{8.37}$$

resulting:

$$\wp = -\varrho c^2. \tag{8.38}$$

The astonishing result of a negative mass density ϱ provided by (8.36), Λ being assumed to be positive, suggests that the cosmological constant, due to the *extra* space-time dimensions, plays the role of a repulsive gravitational source, which is responsible of universe expansion, together with the positive pressure density $\wp$ given by (8.37).

The mass-energy density ϱ and $\wp$ represent the mass-energy and pressure densities of the empty extended space-time V^{16} (*vacuum* energy and pressure) which are seen as matter contributions by an observer living in V^4.

The matter term includes:

1. The mass-energy and pressure densities of matter/interaction fields $(\varrho^{[f]}, \wp^{[f]})$ embedded in V^{16} space-time geometry, as evaluated respect to the reduced connection $\overline{\boldsymbol{\Gamma}}$, being equal to the mass-energy and pressure densities of matter/interaction fields as observable in V^4;

2. The usual V^4 *vacuum energy* $\varrho_{\text{vac}}^{<4>}$ and a *vacuum pressure* $\wp_{\text{vac}}^{<4>}$ densities owed to the cosmological constant (*standard dark energy*);

3. The residual *vacuum energy* $\varrho_{\text{vac}}^{<ex>}$ and a *vacuum pressure* $\wp_{\text{vac}}^{<ex>}$ densities owed to the *extra* space dimensions.

4. The *extra* mass-energy $\varrho_{\text{mat}}^{<ex>}$ and pressure $\wp_{\text{mat}}^{<ex>}$ densities owed to the difference between the usual V^4 connection $\boldsymbol{\Gamma}_{<4>}$ and the reduced connection $\overline{\boldsymbol{\Gamma}}$, previously suggested as hypothetical responsible of *dark matter* (see §6.3.1 in chapter 6):

$$\begin{aligned} \kappa\,\varrho_{\text{mat}}^{<ex>}\,c^2\,g_{00} &= R_{00}^{<4>} - \overline{R}_{00} - \tfrac{1}{2}\left(R^{<4>} - \overline{R}\right)g_{00}, \\ \kappa\,\wp_{\text{mat}}^{<ex>}\,g_{jk} &= R_{jk}^{<4>} - \overline{R}_{jk} - \tfrac{1}{2}\left(R^{<4>} - \overline{R}\right)g_{jk}. \end{aligned} \tag{8.39}$$

Eventually eqs (8.29) and (8.30) may be written equivalently as:

$$\begin{aligned} 3\,\frac{\dot a(t)^2}{c^2\,a(t)^2} &= \kappa\left(\varrho_{\text{vac}}^{<4>} + \varrho_{\text{vac}}^{<ex>} + \varrho_{\text{mat}}^{<ex>} + \varrho^{[f]}\right)c^2|c_g|^2, \\ -2\,\frac{\ddot a(t)}{c^2\,a(t)} - \frac{\dot a^2(t)}{c^2\,a(t)^2} &= \kappa\left(\wp_{\text{vac}}^{<4>} + \wp_{\text{vac}}^{<ex>} + \wp_{\text{mat}}^{<ex>} + \wp^{[f]}\right)|c_g|^2. \end{aligned} \tag{8.40}$$

where:

$$\begin{aligned} &\kappa\,\varrho_{\text{vac}}^{<4>}c^2 = \Lambda, \qquad \kappa\,\wp_{\text{vac}}^{<4>} = -\Lambda, \\ &\kappa\left(\varrho_{\text{vac}}^{<ex>} + \varrho_{\text{mat}}^{<ex>}\right)c^2 = -\tfrac{34}{35}\Lambda - \kappa\,\varrho^{[f]}\,c^2, \\ &\kappa\left(\wp_{\text{vac}}^{<ex>} + \wp_{\text{mat}}^{<ex>}\right) = \tfrac{34}{35}\Lambda + \kappa\,\wp^{[f]}. \end{aligned} \tag{8.41}$$

Remarkably the total mass-energy and pressure densities:

$$\varrho = \varrho_{\text{vac}}^{<4>} + \varrho_{\text{vac}}^{<ex>} + \varrho_{\text{mat}}^{<ex>} + \varrho^{[f]}, \quad \wp = \wp_{\text{vac}}^{<4>} + \wp_{\text{vac}}^{<ex>} + \wp_{\text{mat}}^{<ex>}\,\wp^{[f]}, \tag{8.42}$$

are constant and directly proportional to the cosmological constant.

In the following sections we evaluate the mass-energy and pressure contributions of the matter/interaction fields and the *dark matter* owed to the *extra* dimensions.

8.3.3 The Solution for the Potentials $a_{\bar{\mu}}^{(\bar{\sigma})}$

The potentials $a_{\bar{\mu}}^{(\bar{\sigma})}$ are the eigenvectors of the metric tensor $g_{\bar{\mu}\bar{\nu}}$.

Then they appear to be easily obtained in correspondence to the diagonal solution to the metric (8.7) as:

$$a_{\bar{\mu}}^{(\bar{\sigma})} = c_g \delta_{(0)}^{(\bar{\sigma})} \delta_{\bar{\mu}}^{(0)} + c_g a(t) \Big[\delta_{(1)}^{(\bar{\sigma})} \delta_{\bar{\mu}}^{(1)} + r \delta_{(2)}^{(\bar{\sigma})} \delta_{\bar{\mu}}^{(2)} + \\ + r \sin\theta \delta_{(3)}^{(\bar{\sigma})} \delta_{\bar{\mu}}^{(3)} \Big] + c_{[\underline{s}]} \delta_{(\underline{s})}^{(\bar{\sigma})} \delta_{\bar{\mu}}^{(\underline{s})}. \tag{8.43}$$

We observe that the latter is the most general *real valued* representation. But even a more general *complex* representation is allowed (see §5.2.1 in chapter 5), which is physically relevant providing oscillating wave solutions. In fact the metric tensor is not altered if we introduce imaginary exponential factors into (8.43) which do not affect the products $\eta_{(\bar{\sigma})(\bar{\tau})} a_{\bar{\mu}}^{*(\bar{\sigma})} a_{\bar{\nu}}^{(\bar{\tau})}$ appearing in $g_{\bar{\mu}\bar{\nu}}$, being:

$$\eta_{(\bar{\sigma})(\bar{\tau})} a_{\bar{\mu}}^{(\bar{\sigma})*} a_{\bar{\nu}}^{(\bar{\tau})} = \eta_{(\bar{\sigma})(\bar{\tau})} c_{\bar{\mu}}^{(\bar{\sigma})*} c_{\bar{\nu}}^{(\bar{\tau})}. \tag{8.44}$$

In particular we are interested in distinguishing the *boson* interaction fields $a_{\mu}^{(\bar{\sigma})}$, and the *fermion* matter fields, *i.e.*:

$$a_{\mu}^{(\bar{\sigma})} = c_g \delta_{(0)}^{(\bar{\sigma})} \delta_{\mu}^{(0)} e^{ik_{\bar{\alpha}}^{[0]} x^{\bar{\alpha}}} + c_g a(t) \Big[\delta_{(1)}^{(\bar{\sigma})} \delta_{\mu}^{(1)} e^{ik_{\bar{\alpha}}^{[1]} x^{\bar{\alpha}}} + \\ + r \delta_{(2)}^{(\bar{\sigma})} \delta_{\mu}^{(2)} e^{ik_{\bar{\alpha}}^{[2]} x^{\bar{\alpha}}} + r \sin\theta \delta_{(3)}^{(\bar{\sigma})} \delta_{\mu}^{(3)} e^{ik_{\bar{\alpha}}^{[3]} x^{\bar{\alpha}}} \Big], \tag{8.45}$$

$$a_{\underline{l}}^{(\bar{\sigma})} = c_{[\underline{s}]} a(t) \delta_{(\underline{s})}^{(\bar{\sigma})} \delta_{\underline{l}}^{(\underline{s})} e^{ik_{\alpha}^{[\underline{s}\,\underline{l}]} x^{\alpha}} e^{ik_{\underline{l}}^{[\underline{s}\,\underline{l}]} x^{\underline{l}}}. \tag{8.46}$$

According to the rule established in chapter 7 we may collect the 192 components $a_{\underline{l}}^{(\bar{\sigma})}$ of the vector potential into $192/4 = 48$ Dirac spinors ψ_A, where we may introduce for the sake of simplicity the dummy index $A = 1, 2, \cdots, 48$.

Now we choose:

$$\boldsymbol{k}_{\alpha}^{[A]} = k_{[\alpha]}^{[A]}\gamma_{\alpha}. \tag{8.47}$$

where γ_α are the Dirac *gamma matrices,* which obey the anti-commutation rule:

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}\boldsymbol{I}, \tag{8.48}$$

$\boldsymbol{I}$ being the 4×4 identity matrix. And we write (8.46) in a compact spinor form as:

$$\psi_A = \boldsymbol{c}_A a(t) \exp\left[i\left(k_{[\alpha]}^{[A]}\gamma_\alpha x^\alpha + k_{\underline{l}}^{[A]}x^{\underline{l}}\boldsymbol{I}\right)\right], \tag{8.49}$$

$\boldsymbol{c}_A$ being a constant spinor.

Some comment is required to the result provided by (8.46). In fact, as one is able to realize soon, only the gravitational field components $a_\mu^{(\sigma)}, \sigma, \mu = 0, 1, 2, 3$, can contribute to the vector potentials in V^4, in order to preserve a diagonal metric.

Therefore neither *electro-weak* nor *strong interaction* vector potentials appear in V^4 in the cosmological metric, while their energy does. Non-vanishing $a_\mu^{(\bar{s})}$ components may appear if a co-ordinate change is performed, such that the metric tensor becomes no longer diagonal. In fact while the strength fields $f_{\bar{\mu}\bar{\nu}}^{(\underline{s})}$ of the non-gravitational interaction fields have non-vanishing components $f_{\mu\nu}^{(\underline{s})}$ which are observable in V^4, the vector potential components $a_\mu^{(\underline{s})}$, are zero being necessarily $\mu \neq s$ (being $\mu = 0, 1, 2, 3$, while $s = 4, 5, \cdots, 15$ in $\delta_\mu^{(\underline{s})}$) in order to ensure a diagonal metric. So the vector potentials of the *electro-weak* and *strong interactions* cannot appear within V^4 in a diagonal representation of $g_{\bar{\mu}\bar{\nu}}$.

On the contrary, the *electro-weak* and *strong interactions* vector potential components $a_\mu^{(\underline{s})}$ appear naturally when we perform a co-ordinate transformation, so that the representation of the metric tensor results to be no longer diagonal. Let us consider a general co-ordinate transformation:

$$\hat{x}^{\bar{\mu}} \equiv \hat{x}^{\bar{\mu}}(x^{\bar{\nu}}), \tag{8.50}$$

so that the components of any vector transform as the differentials of the co-ordinates:

$$\mathrm{d}\hat{x}^{\bar{\mu}} = S^{\bar{\mu}}{}_{\bar{\nu}}\,\mathrm{d}x^{\bar{\nu}}, \qquad S^{\bar{\mu}}{}_{\bar{\nu}} = \frac{\partial \hat{x}^{\bar{\mu}}}{\partial x^{\bar{\nu}}}. \tag{8.51}$$

Now each vector potential transforms as:

$$\hat{a}^{(\bar{\sigma})}_{\bar{\mu}} = S_{\bar{\mu}}{}^{\bar{\nu}}\, a^{(\bar{\sigma})}_{\bar{\nu}}. \tag{8.52}$$

In particular it results that the observable components of all the interaction fields:

$$\hat{a}^{(\underline{s})}_{\mu} = S_{\mu}{}^{\bar{\nu}}\, a^{(\underline{s})}_{\bar{\nu}}, \tag{8.53}$$

which become:

$$\hat{a}^{(\underline{s})}_{\mu} = a(t)\,\delta^{(\underline{s})}_{\bar{\nu}}\, S_{\mu}{}^{\bar{\nu}}\, e^{ik^{[\underline{s}\,\bar{\mu}]}_{\bar{\alpha}} x^{\bar{\alpha}}}, \tag{8.54}$$

will be generally non zero.

A similar occurrence was already known in special relativity according to which a magnetic field appears as generated by an electric field, when this latter is observed by a traveling frame. Here the result is generalized to the *electro-weak* and *strong fields,* which appear as generated by a *gravitational field* when it is observed in a different frame.

8.3.4 The Solutions for the Fields $f^{(\bar{\sigma})}_{\bar{\mu}\bar{\nu}}$

Starting form (8.46) we can evaluate the field strength tensors $f^{(\bar{\sigma})}_{\bar{\mu}\bar{\nu}}$.

We have:

$$f^{(0)}_{\bar{\mu}\bar{\nu}} = i\Big(k^{[0]}_{\bar{\mu}} a^{(0)}_{\bar{\nu}} - k^{[0]}_{\bar{\nu}} a^{(0)}_{\bar{\mu}}\Big)\,, \tag{8.55}$$

$$f^{(1)}_{\bar{\mu}\bar{\nu}} = i\Big(k^{[1]}_{\bar{\mu}} a^{(1)}_{\bar{\nu}} - k^{[1]}_{\bar{\nu}} a^{(1)}_{\bar{\mu}}\Big) + \tfrac{1}{c}\tfrac{\dot{a}(t)}{a(t)}\Big(a^{(1)}_{\bar{\mu}}\delta^0_{\bar{\nu}} - a^{(1)}_{\bar{\nu}}\delta^0_{\bar{\mu}}\Big) + \tag{8.56}$$

$$+c^{(1)}_{(2)(3)}\big(a^{(2)}_{\bar{\mu}} a^{(3)}_{\bar{\nu}} - a^{(3)}_{\bar{\mu}} a^{(2)}_{\bar{\nu}}\big),$$

$$f^{(2)}_{\bar{\mu}\bar{\nu}} = i\Big(k^{[2]}_{\bar{\mu}} a^{(2)}_{\bar{\nu}} - k^{[2]}_{\bar{\nu}} a^{(2)}_{\bar{\mu}}\Big) + \tag{8.57}$$

$$+\tfrac{1}{c}\tfrac{\dot{a}(t)}{a(t)}\Big(a^{(2)}_{\bar{\mu}}\delta^0_{\bar{\nu}} - a^{(2)}_{\nu}\delta^0_{\bar{\mu}}\Big) + \tfrac{1}{r}\Big(a^{(2)}_{\bar{\mu}}\delta^1_{\bar{\nu}} - a^{(2)}_{\bar{\nu}}\delta^1_{\bar{\mu}}\Big) + \tag{8.58}$$

$$+c^{(2)}_{(3)(1)}\big(a^{(3)}_{\bar{\mu}} a^{(1)}_{\bar{\nu}} - a^{(1)}_{\bar{\mu}} a^{(3)}_{\bar{\nu}}\big),$$

$$f^{(3)}_{\bar{\mu}\bar{\nu}} = i\Big(k^{[3]}_{\bar{\mu}} a^{(3)}_{\bar{\nu}} - k^{[3]}_{\bar{\nu}} a^{(3)}_{\bar{\mu}}\Big) + \tfrac{1}{c}\tfrac{\dot{a}(t)}{a(t)}\Big(a^{(3)}_{\bar{\mu}}\delta^0_{\bar{\nu}} - a^{(3)}_{\bar{\nu}}\delta^0_{\bar{\mu}}\Big) + \tag{8.59}$$

$$+\tfrac{1}{r}\Big(a^{(3)}_{\bar{\mu}}\delta^1_{\bar{\nu}} - a^{(3)}_{\bar{\nu}}\delta^1_{\bar{\mu}}\Big) + \cot\theta\Big(a^{(3)}_{\bar{\mu}}\delta^2_{\bar{\nu}} - a^{(3)}_{\bar{\nu}}\delta^2_{\bar{\mu}}\Big) + \tag{8.60}$$

$$+c^{(3)}_{(1)(2)}\big(a^{(1)}_{\bar{\mu}} a^{(2)}_{\bar{\nu}} - a^{(2)}_{\bar{\mu}} a^{(1)}_{\bar{\nu}}\big),$$

$$f^{(\underline{s})}_{\bar{\mu}\bar{\nu}} = \Big(ik^{[\underline{s}]}_{\bar{\mu}} + \tfrac{1}{c}\delta^0_{\bar{\mu}}\tfrac{\dot{a}(t)}{a(t)}\Big)a^{(\underline{s})}_{\bar{\nu}} - \Big(ik^{[\underline{s}]}_{\bar{\nu}} + \tfrac{1}{c}\delta^0_{\bar{\nu}}\tfrac{\dot{a}(t)}{a(t)}\Big)a^{(\underline{s})}_{\bar{\mu}} + \tag{8.61}$$

$$+C^{(\underline{s})}_{(q)(r)}\big(a^{(q)}_{\bar{\mu}} a^{(r)}_{\bar{\nu}} - a^{(r)}_{\bar{\mu}} a^{(q)}_{\bar{\nu}}\big).$$

(See chapter 7 for the conventions about the notations of the $C^{(\underline{s})}_{(\underline{q})(\underline{r})}$).

Here we are especially interested in the $f^{(\underline{s})}_{\bar{\mu}\bar{\nu}}$, which represent the only strength tensors appearing in the energy-momentum tensor of the non-gravitational fields. In terms of spinors it results convenient (as we will see in §8.4.2) to introduce:

$$\boldsymbol{F}^{A}_{\bar{\mu}} = a(t)\left[\left(\delta^{\mu}_{\bar{\mu}}\psi^{A}_{,\mu} + \boldsymbol{C}^{A}_{\mu B}\psi^{B}\right) + \delta^{j}_{\bar{\mu}}\,k^{[A]}_{\underline{j}}\psi^{A}\right] + \tfrac{1}{c}\delta^{0}_{\bar{\mu}}\,\dot{a}(t)\psi^{A}, \qquad (8.62)$$

with the symbolic notation $\boldsymbol{C}^{A}_{\bar{\mu}B}$ which collects into spinors the terms $C^{(\underline{s})}_{(q)(r)}\,a^{(q)}_{\bar{\mu}}$.

From (8.62) we obtain:

$$\boldsymbol{F}^{A}_{\bar{\mu}} = a(t)\left(i k^{[A]}_{[\mu]}\,\delta^{\mu}_{\bar{\mu}}\gamma_{\mu}\psi^{A} + \boldsymbol{C}^{A}_{\bar{\mu}B}\psi^{B} + \delta^{j}_{\bar{\mu}}\,k^{[A]}_{\underline{j}}\psi^{A}\right) + \tfrac{1}{c}\delta^{0}_{\bar{\mu}}\,\dot{a}(t)\psi^{A}. \quad (8.63)$$

It is important to remember that ψ^{A} is a scalar respect to co-ordinate transformation in V^4, its elements being built only by $a^{(\underline{s})}_{\underline{l}}$, so its ordinary derivative is covariant in the four-dimensional physical space-time V^4.

8.4 The Energy-Momentum Tensor

In this section we evaluate the energy-momentum tensor $T^{[\underline{s}]}_{\mu\nu}$ in V^4 of each non-gravitational field of strength $f^{(\underline{s})}_{\bar{\mu}\bar{\nu}}$ both by a direct calculation starting from the component representation and following the spinor formulation.

8.4.1 Component Representation

The energy-momentum tensor of each non-gravitational field labelled by $^{(\underline{s})}$ is given by:

$$\kappa T^{[\underline{s}]}_{\mu\nu} = f^{(\underline{s})}_{\bar{\alpha}\mu}\,f^{\bar{\alpha}}_{(\underline{s})\nu} - \tfrac{1}{4}\,f^{(\underline{s})}_{\bar{\alpha}\bar{\beta}}\,f^{\bar{\alpha}\bar{\beta}}_{(\underline{s})}\,g_{\mu\nu} \qquad \text{(non sum over } \underline{s}\text{).} \qquad (8.64)$$

So we have to calculate the tensor products $f^{(\underline{s})}_{\bar\alpha\mu}\, f^{\bar\alpha}_{(\underline{s})_\nu}$ and $f^{(\underline{s})}_{\bar\alpha\bar\beta}\, f^{\bar\alpha\bar\beta}_{(\underline{s})}$.

We observe soon that the non-Abelian terms do not contribute to the energy-momentum tensor, since their products cancel. In fact if we examine, *e.g.*:

$$g^{\bar\alpha\bar\beta} C^{(1)}_{(a)(b)} C_{(1)(c)(d)}\, a^{(a)}_{\bar\mu}\, a^{(b)}_{\bar\alpha}\, a^{(c)}_{\bar\beta}\, a^{(d)}_{\bar\nu}, \qquad a, b, c, d = 1, 2, 3,$$

we obtain:

$$\begin{aligned} &g^{\bar\alpha\bar\beta} C^{(1)}_{(a)(b)} C_{(1)(c)(d)}\, a^{(a)}_{\bar\mu}\, a^{(b)}_{\bar\alpha}\, a^{(c)}_{\bar\beta}\, a^{(d)}_{\bar\nu} = \\ &= g^{\bar\alpha\bar\beta} C^{(1)}_{(2)(3)} C_{(1)(2)(3)} \big(a^{(2)}_{\bar\mu} a^{(3)}_{\bar\alpha} - a^{(3)}_{\bar\mu} a^{(2)}_{\bar\alpha}\big)\big(a^{(2)}_{\bar\beta} a^{(3)}_{\bar\nu} - a^{(2)}_{\bar\beta} a^{(3)}_{\bar\nu}\big). \end{aligned} \tag{8.65}$$

The structure constant (see §6.3.2 and §6.3.3 in chapter 6) are completely *anti*-symmetric and can be written, in general, as:

$$C_{(a)(b)(c)} = g_{[f]}\, \epsilon_{(a)(b)(c)}, \tag{8.66}$$

$g_{[1]}$ being the coupling constant of the field labelled by $^{(1)}$ it follows:

$$\begin{aligned} g^{\bar\alpha\bar\beta} C^{(1)}_{(a)(b)} C_{(1)(c)(d)}\, a^{(a)}_{\bar\mu}\, a^{(b)}_{\bar\alpha}\, a^{(c)}_{\bar\beta}\, a^{(d)}_{\bar\nu} \equiv \\ \equiv -g_{[1]}\, g^{\bar\alpha\bar\beta} a^{(2)}_{\bar\mu} a^{(3)}_{\bar\alpha} a^{(2)}_{\bar\beta} a^{(3)}_{\bar\nu} + g_{[1]}\, g^{\bar\alpha\bar\beta} a^{(2)}_{\bar\mu} a^{(3)}_{\bar\alpha} a^{(2)}_{\bar\beta} a^{(3)}_{\bar\nu} + \\ + g_{[1]}\, g^{\bar\alpha\bar\beta} a^{(3)}_{\bar\mu} a^{(2)}_{\bar\alpha} a^{(2)}_{\bar\beta} a^{(3)}_{\bar\nu} - g_{[1]}\, g^{\bar\alpha\bar\beta} a^{(3)}_{\bar\mu} a^{(2)}_{\bar\alpha} a^{(2)}_{\bar\beta} a^{(3)}_{\bar\nu} = 0. \end{aligned} \tag{8.67}$$

Similar result is obtained for different values of the indices. Then we may drop non-Abelian contributions in evaluating the products appearing in the energy-momentum tensor. We begin evaluating (no sum over $\underline{s}$ will be intended also in what follows):

$$\begin{aligned} &f^{(\underline{s})*}_{\bar\alpha\bar\beta}\, f^{\bar\alpha\bar\beta}_{(\underline{s})} \equiv \eta_{(\underline{s})(\underline{s})}\, g^{\bar\mu\bar\nu} g^{\bar\alpha\bar\beta} f^{(\underline{s})*}_{\bar\mu\bar\alpha} f^{(\underline{s})}_{\bar\nu\bar\beta} = \\ &= 2\Big[k^{\bar\alpha}_{[\underline{s}]} k^{[\underline{s}]}_{\bar\alpha} + \tfrac{1}{c^2} \tfrac{\dot a(t)^2}{a(t)^2}\Big] |c_{[\underline{s}]}|^2 \equiv 2\Big(k^{\bar\alpha}_{[\underline{s}]} k^{[\underline{s}]}_{\bar\alpha} + \tfrac{1}{105}\Lambda\Big)|c_{[\underline{s}]}|^2. \end{aligned} \tag{8.68}$$

Since any wave-particle in V^{16} travels at the speed of light, so that its momentum $p_{\bar{\alpha}}$ is a light-like vector, *i.e.*:

$$p^{\bar{\alpha}} p_{\bar{\alpha}} = 0, \tag{8.69}$$

and because of the Einstein-Planck-De Broglie relation $p_{\bar{\alpha}} = \hbar k_{\bar{\alpha}}$, also the wave vector $k_{\bar{\alpha}}$ results to be light-like, then resulting:

$$k^{\bar{\alpha}} k_{\bar{\alpha}} = 0. \tag{8.70}$$

It follows in (8.68):

$$f^{(\underline{s})*}_{\bar{\alpha}\bar{\beta}} f^{\bar{\alpha}\bar{\beta}}_{(\underline{s})} = \tfrac{2}{105}\Lambda. \tag{8.71}$$

Moreover we need the contributions:

$$\begin{aligned}
&f^{(\underline{s})*}_{\bar{\alpha}0} f^{\bar{\alpha}}_{(\underline{s})0} \equiv \eta_{\underline{s}\underline{s}} g^{\bar{\alpha}\bar{\beta}} f^{(\underline{s})}_{\bar{\alpha}0} f^{(\underline{s})}_{\bar{\beta}0} = \\
&= \left(k^{[\underline{s}]}_0 k^0_{[\underline{s}]} + \tfrac{1}{c^2}\tfrac{\dot{a}(t)^2}{a(t)^2}\right)|c_{[\underline{s}]}|^2 \equiv \left(k^{[\underline{s}]}_0 k^0_{[\underline{s}]} + \tfrac{1}{105}\Lambda\right)|c_{[\underline{s}]}|^2, \\
&f^{(\underline{s})*}_{\bar{\alpha}1} f^{\bar{\alpha}}_{(\underline{s})1} \equiv \eta_{\underline{s}\underline{s}} g^{\bar{\alpha}\bar{\beta}} f^{(\underline{s})}_{\bar{\alpha}1} f^{(\underline{s})}_{\bar{\beta}1} = -k^{[\underline{s}]}_1 k^1_{[\underline{s}]} |c_{[\underline{s}]}|^2 a(t)^2, \\
&f^{(\underline{s})*}_{\bar{\alpha}2} f^{\bar{\alpha}}_{(\underline{s})2} \equiv \eta_{\underline{s}\underline{s}} g^{\bar{\alpha}\bar{\beta}} f^{(\underline{s})}_{\bar{\alpha}2} f^{(\underline{s})}_{\bar{\beta}2} = -k^{[\underline{s}]}_2 k^2_{[\underline{s}]} |c_{[\underline{s}]}|^2 a(t)^2 r^2, \\
&f^{(\underline{s})*}_{\bar{\alpha}3} f^{\bar{\alpha}}_{(\underline{s})3} \equiv \eta_{\underline{s}\underline{s}} g^{\bar{\alpha}\bar{\beta}} f^{(\underline{s})}_{\bar{\alpha}3} f^{(\underline{s})}_{\bar{\beta}3} = -k^{[\underline{s}]}_3 k^3_{[\underline{s}]} |c_{[\underline{s}]}|^2 a(t)^2 r^2 \sin^2\theta.
\end{aligned} \tag{8.72}$$

The following non-vanishing energy-momentum tensor components in V^4 result to be given by:

$$\kappa T^{[\underline{s}]}_{00} = \left(k^{[\underline{s}]}_0 k^0_{[\underline{s}]} + \tfrac{1}{210}\Lambda\right)|c_{[\underline{s}]}|^2, \tag{8.73}$$

$$\kappa T^{[\underline{s}]}_{11} = -\left(k^{[\underline{s}]}_1 k^1_{[\underline{s}]} - \tfrac{1}{210}\Lambda\right)|c_{[\underline{s}]}|^2 a(t)^2, \tag{8.74}$$

$$\kappa T^{[\underline{s}]}_{22} = -\left(k^{[\underline{s}]}_2 k^2_{[\underline{s}]} - \tfrac{1}{210}\Lambda\right)|c_{[\underline{s}]}|^2 a(t)^2 r^2, \tag{8.75}$$

$$\kappa T^{[\underline{s}]}_{33} = -\left(k^{[\underline{s}]}_3 k^3_{[\underline{s}]} - \tfrac{1}{210}\Lambda\right)|c_{[\underline{s}]}|^2 a(t)^2 r^2 \sin^2\theta. \tag{8.76}$$

Comparison with (8.33) leads to the mass-energy and pressure densities of each interaction field of strength $f^{(\underline{s})}_{\mu\nu}$ in V^4, and imposes isotropy of space leading to the conditions:

$$k^{[\underline{s}]}_1 = k^{[\underline{s}]}_2 = k^{[\underline{s}]}_3. \tag{8.77}$$

Introducing the notation:

$$\omega_{[\underline{s}]} = k^0_{[\underline{s}]} c, \tag{8.78}$$

and taking into account (8.70) we have:

$$k^{[\underline{s}]}_1 k^1_{[\underline{s}]} = k^{[\underline{s}]}_2 k^2_{[\underline{s}]} = k^{[\underline{s}]}_3 k^3_{[\underline{s}]} \equiv -\tfrac{1}{3}\left(\tfrac{\omega^2_{[\underline{s}]}}{c^2} + k^{[\underline{s}]}_{\underline{i}} k^{\underline{i}}_{[\underline{s}]}\right). \tag{8.79}$$

And being:

$$k^{[\underline{s}]}_{\underline{i}} k^{\underline{i}}_{[\underline{s}]} = -\tfrac{m^2_{[\underline{s}]} c^2}{\hbar^2}, \tag{8.80}$$

we may write:

$$k^{[\underline{s}]}_1 k^1_{[\underline{s}]} = k^{[\underline{s}]}_2 k^2_{[\underline{s}]} = k^{[\underline{s}]}_3 k^3_{[\underline{s}]} \equiv -\tfrac{1}{3}\left(\tfrac{\omega^2_{[\underline{s}]}}{c^2} - \tfrac{m^2_{[\underline{s}]} c^2}{\hbar^2}\right). \tag{8.81}$$

Eventually we arrive at:

$$\kappa T^{[\underline{s}]}_{00} = \tfrac{1}{c^2}\left(\omega^2_{[\underline{s}]} + \tfrac{1}{210}\Lambda c^2\right)|c_{[\underline{s}]}|^2, \tag{8.82}$$

$$\kappa T^{[\underline{s}]}_{11} = \tfrac{1}{c^2}\left[\tfrac{1}{3}\left(\omega^2_{[\underline{s}]} - \tfrac{m^2_{[\underline{s}]} c^4}{\hbar^2}\right) + \tfrac{1}{210}\Lambda c^2\right]|c_{[\underline{s}]}|^2 a(t)^2, \tag{8.83}$$

$$\kappa T^{[\underline{s}]}_{22} = \tfrac{1}{c^2}\left[\tfrac{1}{3}\left(\omega^2_{[\underline{s}]} - \tfrac{m^2_{[\underline{s}]} c^4}{\hbar^2}\right) + \tfrac{1}{210}\Lambda c^2\right]|c_{[\underline{s}]}|^2 a(t)^2 r^2, \tag{8.84}$$

$$\kappa T^{[\underline{s}]}_{33} = \tfrac{1}{c^2}\left[\tfrac{1}{3}\left(\omega^2_{[\underline{s}]} - \tfrac{m^2_{[\underline{s}]} c^4}{\hbar^2}\right) + \tfrac{1}{210}\Lambda c^2\right]|c_{[\underline{s}]}|^2 a(t)^2 r^2 \sin^2\theta. \tag{8.85}$$

Then we can identify the mass-energy and pressure densities as:

$$\varrho_{[\underline{s}]}\, c^2 = \tfrac{1}{\kappa c^2}\Big(\omega_{[\underline{s}]}^2 + \tfrac{1}{210}\Lambda c^2\Big), \tag{8.86}$$

$$\wp_{[\underline{s}]} = \tfrac{1}{\kappa c^2}\Big[\tfrac{1}{3}\Big(\omega_{[\underline{s}]}^2 - \tfrac{m_{[\underline{s}]}^2 c^4}{\hbar^2}\Big) + \tfrac{1}{210}\Lambda c^2\Big]. \tag{8.87}$$

The previous results are easily extended to a many and even infinite particles solution provided that we replace the wave number vectors $k^{\alpha}_{[\underline{s}]}$ with diagonal matrices:

$$\mathsf{K}^{[\underline{s}]}_{\bar{\alpha}} \equiv \begin{pmatrix} k^{[\underline{s}]}_{[\mathbf{1}]\bar{\alpha}} & 0 & \cdots & 0 \\ 0 & k^{[\underline{s}]}_{[\mathbf{2}]\bar{\alpha}} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \tag{8.88}$$

The vector potential solutions are also given by matrices, becoming:

$$\mathsf{a}^{(\underline{s})}_{\bar{\mu}} = a(t)\,\delta^{(\underline{s})}_{\bar{\mu}}\, c_{[\underline{s}]}\, e^{i\mathsf{K}^{[\underline{s}\,\mu]}_{\bar{\alpha}} x^{\bar{\alpha}}}, \tag{8.89}$$

from which the field strength tensor matrices result:

$$\begin{aligned} \mathsf{f}^{(\underline{s})}_{\bar{\mu}\bar{\nu}} &= -i\left(\mathsf{K}^{[\underline{s}]}_{\bar{\mu}}\delta^{(\underline{s})}_{\bar{\nu}} - \mathsf{K}^{[\underline{s}]}_{\bar{\nu}}\delta^{(\underline{s})}_{\bar{\mu}}\right) a(t)\, c_{[\underline{s}]}\, e^{i\mathsf{K}^{[\underline{s}]}_{\bar{\alpha}} x^{\bar{\alpha}}} + \\ &+ \tfrac{1}{c}\,\dot{a}(t)\Big(\delta^{(\underline{s})}_{\bar{\mu}}\delta^{0}_{\bar{\nu}} - \delta^{(\underline{s})}_{\bar{\nu}}\delta^{0}_{\bar{\mu}}\Big) c_{[\underline{s}]}\, e^{i\mathsf{K}^{[\underline{s}]}_{\bar{\alpha}} x^{\bar{\alpha}}}. \end{aligned} \tag{8.90}$$

So mass-energy and pressure densities of the fields $\mathsf{f}^{(\underline{s})}_{\mu\nu}$ are given by:

$$\kappa\,\varrho_{[\underline{s}]}\, c^2 = \mathrm{Tr}\left(\mathsf{K}^{[\underline{s}]}_{0}\mathsf{K}^{0}_{[\underline{s}]}\right) + \tfrac{1}{210}\Lambda, \tag{8.91}$$

$$\kappa\,\wp_{[\underline{s}]} = -\mathrm{Tr}\left(\mathsf{K}^{[\underline{s}]}_{1}\mathsf{K}^{1}_{[\underline{s}]}\right) + \tfrac{1}{210}\Lambda. \tag{8.92}$$

In fact the latter result is equivalent to the summation of each particle contribution:

$$\kappa\,\varrho_{[\underline{s}]}\, c^2 = \textstyle\sum_p \left(k^{[\underline{s}]}_{p0} k^{0}_{p[\underline{s}]}\right) + \tfrac{1}{210}\Lambda, \tag{8.93}$$

$$\kappa p_{[\underline{s}]} = -\textstyle\sum_p \left(k^{[\underline{s}]}_{p1} k^{1}_{p[\underline{s}]}\right) + \tfrac{1}{210}\Lambda. \tag{8.94}$$

8.4.2 Spinor Representation

In order to evaluate the energy-momentum tensor in terms of spinors we can follow two equivalent approaches:

1. The former combines directly the products of the strength field as given by (8.63) in which the derivative of each spinor is explicitly calculated.

2. While the latter combines respectively one of the strength fields as provided by (8.63) with another one as it is given by (8.62).

First Approach

Adopting the representation (8.63) of the field strength in terms of spinors, the energy-momentum tensor of the *fermion* fields assumes the following form:

$$\kappa T^{[A]}_{\mu\nu} = 2\boldsymbol{F}^{A+}_{\mu}\boldsymbol{F}_{A\nu} - \tfrac{1}{2}\boldsymbol{F}^{A+}_{\bar{\alpha}}\boldsymbol{F}^{\bar{\alpha}}_{A}\, g_{\mu\nu} \qquad \text{(no sum over } A\text{)}. \tag{8.95}$$

A factor 2 appears because each product of any $\boldsymbol{F}^{A}_{\bar{\mu}}$ contributes twice to the products of the anti-symmetric strength tensors $f^{(\underline{s})}_{\bar{\mu}\bar{\nu}}$. We have:

$$\boldsymbol{F}^{\bar{\alpha}+}_{A}\boldsymbol{F}^{A}_{\bar{\alpha}} = -\psi^{+}_{A}\Big[\tfrac{1}{2}k^{[A]}_{[\alpha]}k^{[\alpha]}_{[A]}g^{\alpha\beta}\big(\gamma_{\alpha}\gamma_{\beta}+\gamma_{\beta}\gamma_{\alpha}\big) + \\ +\Big(k^{[A]}_{\underline{j}}k^{\underline{j}}_{[A]} + \tfrac{1}{c^2}\tfrac{\dot{a}(t)^2}{a(t)^2}\Big)\boldsymbol{I}\Big]\psi^{A}. \tag{8.96}$$

Taking into account (8.47) we have:

$$\tfrac{1}{2}k^{[A]}_{[\alpha]}k^{[\alpha]}_{[A]}g^{\alpha\beta}\big(\gamma_{\alpha}\gamma_{\beta}+\gamma_{\beta}\gamma_{\alpha}\big) = g^{\alpha\beta}g_{\alpha\beta}k^{[A]}_{[\alpha]}k^{[\alpha]}_{[A]}\boldsymbol{I}. \tag{8.97}$$

Some care is required in evaluating:

$$g^{\alpha\beta} g_{\alpha\beta} k_{[\alpha]}^{[A]} k_{[A]}^{[\alpha]} \equiv g^{00} g_{00} k_{[0]}^{[A]} k_{[A]}^{[0]} + g^{11} g_{11} k_{[1]}^{[A]} k_{[A]}^{[1]} + \\ + g^{22} g_{22} k_{[22]}^{[A]} k_{[A]}^{[22]} + g^{33} g_{33} k_{[33]}^{[A]} k_{[A]}^{[33]} \equiv k_{[\alpha]}^{[A]} k_{[A]}^{[\alpha]}. \tag{8.98}$$

Then it results:

$$\boldsymbol{F}_A^{\alpha+} \boldsymbol{F}_\alpha^A = -\left[k_\alpha^{[A]} k_{[A]}^\alpha + k_{\underline{j}}^{[A]} k_{[A]}^{\underline{j}} + \frac{1}{c^2} \frac{\dot{a}(t)^2}{a(t)^2} \right] \psi_A^+ \psi^A. \tag{8.99}$$

Vector $k_{\bar{\alpha}}^{[A]}$ being light-like, since the particles travel at the speed of light c in V^{16}, so that:

$$k_{\bar{\alpha}}^{[A]} k_{[A]}^{\bar{\alpha}} \equiv k_\alpha^{[A]} k_{[A]}^\alpha + k_{\underline{j}}^{[A]} k_{[A]}^{\underline{j}} = 0 \quad \Longrightarrow \quad k_{[\alpha]}^{[A]} k_{[A]}^{[\alpha]} = \frac{m_A^2 c^2}{\hbar^2}, \tag{8.100}$$

it just results:

$$\boldsymbol{F}_A^{\alpha+} \boldsymbol{F}_\alpha^A = -\frac{1}{c^2} \frac{\dot{a}(t)^2}{a(t)^2} \psi_A^+ \psi^A, \tag{8.101}$$

as expected. Moreover we evaluate:

$$\boldsymbol{F}_\mu^{A+} \boldsymbol{F}_{A\nu} = -\psi_A^+ \left[\frac{1}{2} \left(k^{[A]} \right)^2 \left(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu \right) + \frac{1}{c^2} \delta_\mu^0 \delta_\nu^0 \frac{\dot{a}(t)^2}{a(t)^2} \boldsymbol{I} \right] \psi^A, \tag{8.102}$$

which, thanks (8.47) becomes:

$$\boldsymbol{F}_\mu^{A+} \boldsymbol{F}_{A\nu} = -\left[k_\mu^{[A]} k_{[A]\nu} + \frac{1}{c^2} \delta_\mu^0 \delta_\nu^0 \frac{\dot{a}(t)^2}{a(t)^2} \right] \psi_A^+ \psi^A. \tag{8.103}$$

Then the energy-momentum tensor results:

$$\kappa T_{\mu\nu}^{[A]} = -\left[k_\mu^{[A]} k_{[A]\nu} + \frac{1}{c^2} \delta_\mu^0 \delta_\nu^0 \frac{\dot{a}(t)^2}{a(t)^2} - \frac{1}{2c^2} \frac{\dot{a}(t)^2}{a(t)^2} g_{\mu\nu} \right] \psi_A^+ \psi^A. \tag{8.104}$$

The components of which are:

$$\kappa T_{00}^{[A]} = -\left(k_0^{[A]} k_{[A]}^0 + \frac{1}{210} \Lambda \right) \psi_A^+ \psi^A, \tag{8.105}$$

$$\kappa T_{11}^{[A]} = \left(k_1^{[A]} k_{[A]}^1 - \frac{1}{210} \Lambda \right) \psi_A^+ \psi^A a(t)^2, \tag{8.106}$$

$$\kappa T_{22}^{[A]} = \left(k_2^{[A]} k_{[A]}^2 - \tfrac{1}{210}\Lambda\right)\psi_A^+ \psi^A a(t)^2 r^2, \tag{8.107}$$

$$\kappa T_{33}^{[A]} = \left(k_3^{[A]} k_{[A]}^3 - \tfrac{1}{210}\Lambda\right)\psi_A^+ \psi^A a(t)^2 r^2 \sin^2\theta. \tag{8.108}$$

Or in an equivalent form:

$$\kappa T_{00}^{[A]} = \left(k_0^{[A]} k_{[A]}^0 + \tfrac{1}{210}\Lambda\right) c_{[A]}^* c^{[A]}, \tag{8.109}$$

$$\kappa T_{11}^{[A]} = -\left(k_1^{[A]} k_{[A]}^1 - \tfrac{1}{210}\Lambda\right) |c_{[A]}^* c^{[A]} a(t)^2, \tag{8.110}$$

$$\kappa T_{22}^{[A]} = -\left(k_2^{[A]} k_{[A]}^2 - \tfrac{1}{210}\Lambda\right) |c_{[A]}^* c^{[A]} a(t)^2 r^2, \tag{8.111}$$

$$\kappa T_{33}^{[A]} = -\left(k_3^{[A]} k_{[A]}^3 - \tfrac{1}{210}\Lambda\right) |c_{[A]}^* c^{[A]} a(t)^2 r^2 \sin^2\theta. \tag{8.112}$$

Isotropy condition and analogous calculations as in the previous sections lead to the same results according to a spinor formalism.

Second Approach

A second approach leading to the energy-momentum tensor of the fermion fields in an alternative equivalent form combines the products of the strength fields as given respectively by (8.62) and (8.63).

We obtain the following products:

$$\boldsymbol{F}_{\bar{\alpha}}^{A+} \boldsymbol{F}_A^{\bar{\alpha}} = i k_{[A]}^{[\alpha]} \psi_A^+ \gamma^\alpha \psi_{,\alpha}^A + k_{\underline{j}}^{[A]} k_{[A]}^{\underline{j}} \psi_A^+ \psi^A + \tfrac{2}{105}\Lambda \psi_A^+ \psi^A. \tag{8.113}$$

Introducing the fermion masses m_A we have:

$$k_{\underline{j}}^{[A]} k_{[A]}^{\underline{j}} = -\tfrac{m_A^2 c^2}{\hbar^2}, \qquad k_{[A]}^{[\alpha]} = \tfrac{m_A c}{\hbar}, \tag{8.114}$$

from which it follows:

$$\begin{aligned} \boldsymbol{F}_{\bar{\alpha}}^{A+} \boldsymbol{F}_A^{\bar{\alpha}} &= i k_{[A]}^{[\alpha]} \psi_A^+ \gamma^\alpha \psi_{,\alpha}^A - \tfrac{m_{[A]}^2 c^2}{\hbar^2} \psi_A^+ \psi^A + \tfrac{2}{105}\Lambda \psi_A^+ \psi^A \equiv \\ &\equiv i \tfrac{m_A c}{\hbar} \psi_A^+ \left(\gamma^\alpha \psi_{,\alpha}^A + \tfrac{i m_{[A]} c}{\hbar} \psi^A\right) + \tfrac{2}{105}\Lambda \psi_A^+ \psi^A = \tfrac{2}{105}\Lambda \psi_A^+ \psi^A, \end{aligned} \tag{8.115}$$

thanks to Dirac equation:

$$\gamma^{\alpha}\psi^{A}_{,\alpha} + \tfrac{im_{[A]}c}{\hbar}\psi^{A} = 0. \qquad (8.116)$$

Moreover we need:

$$\boldsymbol{F}^{A+}_{\mu}\boldsymbol{F}_{A\nu} = \tfrac{1}{2}\psi^{+}_{A}\Big(ik^{[A]}_{[\mu]}\gamma_{\mu}\psi^{A}_{,\nu} + ik^{[A]}_{[\nu]}\gamma_{\nu}\psi^{A}_{,\mu}\Big) + \tfrac{1}{105}\varLambda\delta^{0}_{\mu}\delta^{0}_{\nu}\psi^{+}_{A}\psi^{A}, \quad (8.117)$$

which is equivalent to:

$$\boldsymbol{F}^{A+}_{\mu}\boldsymbol{F}_{A\nu} = \tfrac{1}{2}i\tfrac{m_{[A]}c}{\hbar}\psi^{+}_{A}\Big(\gamma_{\mu}\psi^{A}_{,\nu} + \gamma_{\nu}\psi^{A}_{,\mu}\Big) + \tfrac{1}{105}\varLambda\delta^{0}_{\mu}\delta^{0}_{\nu}\psi^{+}_{A}\psi^{A}. \qquad (8.118)$$

Therefore the energy-momentum tensors becomes:

$$\kappa T^{[A]}_{\mu\nu} = i\tfrac{m_{[A]}c}{\hbar}\psi^{+}_{A}\Big(\gamma_{\mu}\psi^{A}_{,\nu}+\gamma_{\nu}\psi^{A}_{,\mu}\Big) + \tfrac{1}{105}\varLambda\Big(\delta^{0}_{\mu}\delta^{0}_{\nu} - \tfrac{1}{2}g_{\mu\nu}\Big)\psi^{+}_{A}\psi^{A}. \quad (8.119)$$

In particular the Hamiltonian density is given by:

$$\kappa\mathcal{H}^{[A]} \equiv \kappa T^{[A]}_{00} = i\tfrac{m_{[A]}c}{\hbar}\psi^{+}_{A}\gamma_{0}\psi^{A}_{,0} + \tfrac{1}{210}\varLambda\psi^{+}_{A}\psi^{A}. \qquad (8.120)$$

Now from Dirac equation we may solve:

$$\gamma_{0}\psi^{A}_{,0} = -\gamma^{i}\psi^{A}_{,i} - \tfrac{im_{[A]}c}{\hbar}\psi^{A}, \qquad (8.121)$$

which we substitute into (8.120) obtaining, at the end:

$$\kappa\mathcal{H} = \tfrac{m_{[A]}c}{\hbar}\psi^{+}_{A}\Big(-i\gamma^{i}\psi^{A}_{,i} + \tfrac{m_{[A]}c}{\hbar}\psi^{A}\Big) + \tfrac{1}{210}\varLambda\psi^{+}_{A}\psi^{A}. \qquad (8.122)$$

8.4.3 Geometry Contributions to Dark Matter and Energy

We are know able to evaluate explicitly the contributions to *dark matter* and *dark energy* arising from the *extra* space dimensions of space-time. We start from the last two eqs in (8.41), *i.e.*:

$$\kappa\Big(\varrho^{<ex>}_{\mathrm{vac}} + \varrho^{<ex>}_{\mathrm{mat}}\Big)c^{2} = -\tfrac{34}{35}\varLambda - \kappa\varrho^{[f]}c^{2}, \qquad (8.123)$$

$$\kappa\Big(\wp_{\text{vac}}^{<ex>} + \wp_{\text{mat}}^{<ex>}\Big) = \tfrac{34}{35}\varLambda + \kappa\,\wp^{[f]}, \tag{8.124}$$

and replace $\varrho^{[f]}\,c^2$, $\wp^{[f]}$, which represent the total mass-energy and pressure densities contributions of all the non-gravitational fields and all particles, with their values, which thanks to (8.86) and (8.87) are given by:

$$\kappa\,\varrho^{[f]}\,c^2 = \tfrac{1}{c^2}\textstyle\sum_{p,\underline{s}}\omega_{[\underline{s}]}^2 + \tfrac{1}{210}\varLambda, \tag{8.125}$$

$$\kappa\,\wp^{[f]} = \tfrac{1}{3}\textstyle\sum_{p,\underline{s}}\Big(\frac{\omega_{[\underline{s}]}^2}{c^2} - \frac{m_{[\underline{s}]}^2 c^2}{\hbar^2}\Big) + \tfrac{1}{210}\varLambda. \tag{8.126}$$

After substitution we obtain:

$$\kappa\Big(\varrho_{\text{vac}}^{<ex>} + \varrho_{\text{mat}}^{<ex>}\Big)c^2 = -\tfrac{41}{42}\varLambda - \tfrac{1}{c^2}\textstyle\sum_{p,\underline{s}}\omega_{[\underline{s}]}^2, \tag{8.127}$$

$$\kappa\Big(\wp_{\text{vac}}^{<ex>} + \wp_{\text{mat}}^{<ex>}\Big) = \tfrac{41}{42}\varLambda + \tfrac{1}{3}\textstyle\sum_{p,\underline{s}}\Big(\frac{\omega_{[\underline{s}]}^2}{c^2} - \frac{m_{[\underline{s}]}^2 c^2}{\hbar^2}\Big). \tag{8.128}$$

We recognize in the terms involving the cosmological constant the *vacuum* contributions arising because of the *extra* space dimensions:

$$\kappa\,\varrho_{\text{vac}}^{<ex>}\,c^2 = -\tfrac{41}{42}\varLambda, \tag{8.129}$$

$$\kappa\,\wp_{\text{vac}}^{<ex>} = \tfrac{41}{42}\varLambda, \tag{8.130}$$

and in the remaining terms:

$$\varrho_{\text{mat}}^{<ex>}\Big)c^2 = \textstyle\sum_{p,\underline{s}}\omega_{[\underline{s}]}^2, \tag{8.131}$$

$$\wp_{\text{mat}}^{<ex>}\Big) = \tfrac{1}{3}\textstyle\sum_{p,\underline{s}}\Big(\frac{\omega_{[\underline{s}]}^2}{c^2} - \frac{m_{[\underline{s}]}^2 c^2}{\hbar^2}\Big), \tag{8.132}$$

the contributions arising from matter-energy residing in the *extra* space.

8.5 Geodesics and Particle Charges

Let us now consider a geodesic path in the extended space-time V^{16} described by the parametric equations:

$$x^{\bar{\mu}} \equiv x^{\bar{\mu}}(t), \tag{8.133}$$

and the tangent vector:

$$u^{\bar{\mu}} = \frac{\mathrm{d}x^{\bar{\mu}}}{\mathrm{d}t}, \qquad \bar{\mu} = 0, 1, \cdots, 15. \tag{8.134}$$

The equation of a *light-like* geodesic in V^{16}, writes:

$$\frac{\mathrm{d}u^{\bar{\mu}}}{\mathrm{d}\check{s}} + \Gamma^{\bar{\mu}}_{\bar{\alpha}\bar{\nu}} u^{\bar{\alpha}} u^{\bar{\nu}} = 0. \tag{8.135}$$

According to the previous results – see chapter 6, eq (6.67) – we know that:

$$\Gamma^{\bar{\mu}}_{\bar{\alpha}\bar{\nu}} = \bar{\gamma}^{\bar{\mu}}_{\bar{\alpha}\bar{\nu}} - \ \tfrac{1}{2} \left(a^{(\underline{s})}_{\bar{\alpha}} f^{\bar{\mu}}_{(\underline{s})\bar{\nu}} + f^{\bar{\mu}}_{(\underline{s})\bar{\alpha}} a^{(\underline{s})}_{\bar{\nu}} \right).$$

From which it follows:

$$\frac{\mathrm{d}u^{\bar{\mu}}}{\mathrm{d}t} = -\bar{\gamma}^{\bar{\mu}}_{\bar{\alpha}\bar{\nu}} u^{\bar{\alpha}} u^{\bar{\nu}} + \ \tfrac{1}{2} \left(a^{(\underline{s})}_{\bar{\alpha}} f^{\bar{\mu}}_{(\underline{s})\bar{\nu}} + f^{\bar{\mu}}_{(\underline{s})\bar{\alpha}} a^{(\underline{s})}_{\bar{\nu}} \right) u^{\bar{\alpha}} u^{\bar{\nu}}. \tag{8.136}$$

And thanks to symmetries:

$$\frac{\mathrm{d}u^{\bar{\mu}}}{\mathrm{d}t} = -\bar{\gamma}^{\bar{\mu}}_{\bar{\alpha}\bar{\nu}} u^{\bar{\alpha}} u^{\bar{\nu}} + u^{\bar{\alpha}} a^{(\underline{s})}_{\bar{\alpha}} f^{\bar{\mu}}_{(\underline{s})\bar{\nu}} u^{\bar{\nu}}. \tag{8.137}$$

Introducing the notations:

$$g^{\bar{\mu}} = -\bar{\gamma}^{\bar{\mu}}_{\bar{\alpha}\bar{\nu}} u^{\bar{\alpha}} u^{\bar{\nu}}, \quad \mathcal{E}^{\bar{\mu}}_{(\underline{s})} = -f^{\bar{\mu}}_{(\underline{s})\bar{\nu}} u^{\bar{\nu}}, \quad e^{(\underline{s})} = -m_{[\underline{s}]} u^{\bar{\alpha}} a^{(\underline{s})}_{\bar{\alpha}}, \tag{8.138}$$

we arrive at the law of wave-particle geodesic motion, related to the partially reduced connection:

$$\frac{\mathrm{d}u^{\bar{\mu}}}{\mathrm{d}t} = g^{\bar{\mu}} + \frac{e^{(\underline{s})}}{m_{[\underline{s}]}} \mathcal{E}^{\bar{\mu}}_{(\underline{s})}, \tag{8.139}$$

where $g^{\bar{\mu}}$ is interpreted as gravity acceleration field (dark matter contribution included), $\mathcal{E}^{\bar{\mu}}_{(\underline{s})}$ as the electric counterpart force of electro-weak and gluon fields, $e^{(\underline{s})}$ as the respective charge and $m_{[\underline{s}]}$ as the traveling wave-particle mass.

In general, $e^{(\underline{s})}$ as defined by (8.138) is not a constant. Actually we know that each wave-particle (*boson* or *fermion*) is represented by a solution for a potential $a^{(\bar{\sigma})}_{\bar{\mu}}$ of the field equations. Then eq (8.138) is to be written, properly, as:

$$e^{(\underline{s})}_{[\bar{\sigma}]} = -\hbar k^{\bar{\alpha}}_{[\bar{\sigma}]} a^{(\underline{s})}_{\bar{\alpha}}, \tag{8.140}$$

since:

$$p^{\bar{\alpha}}_{[\bar{\sigma}]} = m_{[\bar{\sigma}]} u^{\bar{\alpha}}_{[\bar{\sigma}]}, \tag{8.141}$$

and:

$$p^{\bar{\alpha}}_{[\bar{\sigma}]} = \hbar k^{\bar{\alpha}}_{[\bar{\sigma}]}, \tag{8.142}$$

thanks to De Broglie and Einstein-Planck relations. In correspondence to solutions (8.46) the charge becomes:

$$e^{(\underline{s})} = -\hbar k^{\bar{\alpha}}_{[\underline{s}]} c_{[\underline{s}]} \delta^{(\underline{s})}_{\bar{\alpha}} e^{i k^{[\underline{s}\,\bar{\alpha}]}_{\bar{\alpha}} x^{\bar{\alpha}}}. \tag{8.143}$$

Each particle is actually localized (maximum probability of presence) on the wave-front of equation:

$$k^{[\underline{s}\,\bar{\alpha}]}_{\bar{\alpha}} x^{\bar{\alpha}} = 0, \tag{8.144}$$

on which the charge assumes the constant value:

$$e^{(\underline{s})} = -\hbar k^{\bar{\alpha}}_{[\underline{s}]} c_{[\underline{s}]} \delta^{(\underline{s})}_{\bar{\alpha}} \equiv -\hbar k^{\underline{s}}_{[\underline{s}]} c_{[\underline{s}]}. \tag{8.145}$$

We remember that the dimensional constants required to fit the standard systems of measure units (M.K.S. or c.g.s., or other) have been implicitly included within $a^{(\underline{s})}_{\bar{\alpha}}$, $f^{(\underline{s})}_{\bar{\mu}\bar{\nu}}$. Then also the unit of measure of the charges here is not the usual one. Of course, when required it always possible to restore the conventional unit system, introducing suitable dimensional constant factors.

If, *e.g.*, the $\underline{s}$-th *extra*-component of the momentum $p^{\underline{s}}_{[\underline{s}]} = \hbar k^{\underline{s}}_{[\underline{s}]}$ is interpreted as related to the particle rest mass term $m_{[\underline{s}]}$:

$$p^{\underline{s}}_{[\underline{s}]} \equiv \hbar k^{\underline{s}}_{[\underline{s}]} c_{[\underline{s}]} = c_{[\underline{s}]} m_{[\underline{s}]} c, \tag{8.146}$$

the charge becomes:

$$e^{(\underline{s})} = -c_{[\underline{s}]} m_{[\underline{s}]} c. \tag{8.147}$$

Compactification of the extra co-ordinates could lead to the quantization of charges and the consequent well known mass gap (see [25]) could probably be avoided assuming that the compactification path is not a circle but a curve of higher length, as *e.g.*, a fractal path (see [29]).

Chapter 9

A Way to Quantum Gravity?

Abstract
In this last chapter, after quantizing the non-gravitational spinor fields, we show also a way leading to quantization of the gravitational field in correspondence to a cosmological solution. The present approach arises just from the assumption that the fundamental fields (including gravity) are the eigenvectors of the metric tensor in a multidimensional space-time.

9.1 Introduction

In the previous chapter we have evaluated the energy-momentum tensor of the non-gravitational fields $a_{\underline{\mu}}^{(\underline{s})}$, $\underline{s} = 4, 5, \cdots, 15$, in correspondence to the diagonal metric (8.7) which generalizes to V^{16} the cosmological Robertson-Walker metric of the physical space-time V^4. In particular we have found representations both in terms of components and in terms of spinors.

But nothing was said about the energy-momentum of the gravitational field in V^4, the latter field being embodied within the geometry of the space-time manifold.

So also quantization of gravity seems to be prevented at least following a way similar to the electromagnetic or other interaction fields.

In the present chapter we want just to attack the problem of field quantization. First of all we quantize the fermion fields and then, we propose a natural way to quantization of the gravitational field itself.

9.2 Quantization of the Fermion Fields

The diagonal metric solution we examined in the previous chapter 8 allows for the non-gravitational field $a_{\underline{\mu}}^{(\underline{s})}$, only the non-vanishing components $a_{\underline{k}}^{(\underline{s})}$, which are related to fermions.

Quantization can be performed as usual starting from their Hamiltonian densities, equal to the $T_{00}^{[\underline{s}]}$ components of their energy-momentum tensor.

In correspondence to the cosmological solution examined in the previous chapter we have, in presence of more particles – see eqs (8.86) and (8.93)

– the following Hamiltonian density:

$$\mathcal{H}^{[\underline{s}]} = \tfrac{1}{c^2\kappa}\textstyle\sum_p \Big(\omega^2_{[\underline{s}p]} + \tfrac{1}{210}\Lambda c^2\Big)|c_{[\underline{s}]}|^2. \tag{9.1}$$

Then the total Hamiltonian in a region $\mathcal{D}$ of the space, of volume V results by integration of the Hamiltonian density over the volume of the domain $\mathcal{D}$, *i.e.*:

$$\begin{aligned} H^{[\underline{s}]} &= \tfrac{1}{c^2\kappa}\int_{\mathcal{D}} \textstyle\sum_p \Big(\omega^2_{[\underline{s}]p} + \tfrac{1}{210}\Lambda c^2\Big)|c_{[\underline{s}]}|^2 \sqrt{|g^{[3]}|}\, \mathrm{d}^3x \equiv \\ &\equiv \tfrac{V}{c^2\kappa}\Big(\omega^2_{[\underline{s}]p} + \tfrac{1}{210}\Lambda c^2\Big)|c_{[\underline{s}]}|^2. \end{aligned} \tag{9.2}$$

Conveniently we replace the coefficients $c_{[\underline{s}]}$ with a combination of two other coefficients $b_{[\underline{s}]p}, d_{[\underline{s}]p}$ (and their complex conjugates), defined by:

$$\begin{aligned} &\sqrt{\tfrac{V}{c^2\kappa}\Big(\omega^2_{[\underline{s}]p} + \tfrac{1}{210}\Lambda c^2\Big)}\ c_{[\underline{s}]} = \\ &= \sqrt{\tfrac{1}{2}\hbar\omega_{[\underline{s}]p}}\ \Big[\big(b^*_{[\underline{s}]p} + b_{[\underline{s}]p}\big) + i\big(d_{[\underline{s}]p} - d^*_{[\underline{s}]p}\big)\Big], \end{aligned} \tag{9.3}$$

so that the square modulus results:

$$\begin{aligned} &\tfrac{V}{c^2\kappa}\Big(\omega^2_{[\underline{s}p]} + \tfrac{1}{210}\Lambda\Big)|c_{[\underline{s}]}|^2 = \tfrac{1}{2}\hbar\omega_{[\underline{s}]}\Big[\big(b_{[\underline{s}]p}\big)^2 + \big(b^*_{[\underline{s}]p}\big)^2 + \\ &+2b^*_{[\underline{s}]p}b_{[\underline{s}]p} + \big(d_{[\underline{s}]p}\big)^2 + \big(d^*_{[\underline{s}]p}\big)^2 - 2d_{[\underline{s}]p}d^*_{[\underline{s}]p}\Big]. \end{aligned} \tag{9.4}$$

Then the Hamiltonian becomes:

$$\begin{aligned} H^{[\underline{s}]} &= \tfrac{1}{2}\sum_p \hbar\omega_{[\underline{s}p]}\Big[\big(b_{[\underline{s}]p}\big)^2 + \big(b^*_{[\underline{s}]p}\big)^2 + \\ &+2b^*_{[\underline{s}]p}b_{[\underline{s}]p} + \big(d_{[\underline{s}]p}\big)^2 + \big(d^*_{[\underline{s}]p}\big)^2 - 2d_{[\underline{s}]p}d^*_{[\underline{s}]p}\Big]. \end{aligned} \tag{9.5}$$

Quantization is obtained by replacing the arbitrary coefficients $b_{[\underline{s}]}, d_{[\underline{s}]}$ by the creation and annihilation fermion operators:

$$b_{[\underline{s}]p} \longrightarrow \mathsf{b}_{[\underline{s}]p}, \qquad d_{[\underline{s}]p} \longrightarrow \mathsf{d}_{[\underline{s}]p}, \tag{9.6}$$

which fulfill the anti-commutation relations:

$$\{\mathsf{b}_{[\underline{r}]p}, \mathsf{b}_{[\underline{s}]q}\} = 0, \qquad \{\mathsf{d}_{[\underline{r}]p}, \mathsf{d}_{[\underline{s}]q}\} = 0, \tag{9.7}$$

$$\{\mathsf{b}_{[\underline{r}]p}, \mathsf{b}^*_{[\underline{s}]q}\} = \delta_{\underline{r}\,\underline{s}}\delta_{pq}\mathsf{I}, \qquad \{\mathsf{d}_{[\underline{r}]p}, \mathsf{d}^*_{[\underline{s}]q}\} = \delta_{\underline{r}\,\underline{s}}\delta_{pq}\mathsf{I}. \tag{9.8}$$

Because of anti-commutation laws (9.7) each squared operator results to be null and the Hamiltonian operator becomes:

$$\mathsf{H}^{[\underline{s}]} = \sum_p \hbar\omega_{[\underline{s}]p}\Big(\mathsf{b}^+_{[\underline{s}]p}\mathsf{b}_{[\underline{s}]p} - \mathsf{d}_{[\underline{s}]p}\mathsf{d}^+_{[\underline{s}]p}\Big). \tag{9.9}$$

And thanks to (9.8) it becomes:

$$\mathsf{H}^{[\underline{s}]} = \sum_p \hbar\omega_{[\underline{s}]p}\Big(\mathsf{b}^+_{[\underline{s}]p}\mathsf{b}_{[\underline{s}]p} + \mathsf{d}^+_{[\underline{s}]p}\mathsf{d}_{[\underline{s}]p} - \mathsf{I}\Big). \tag{9.10}$$

Summation over $\underline{s}$ gives the total Hamiltonian operator of all the non gravitational fields:

$$\mathsf{H}^{[f]} = \sum_{\underline{s}}\sum_p \hbar\omega_{[\underline{s}]p}\Big(\mathsf{b}^+_{[\underline{s}]p}\mathsf{b}_{[\underline{s}]p} + \mathsf{d}^+_{[\underline{s}]p}\mathsf{d}_{[\underline{s}]p} - \mathsf{I}\Big), \tag{9.11}$$

which is the known Hamiltonian of the quantized Dirac fields.

9.3 Energy-Momentum Tensor of Gravitational Field

In the present section we investigate a way to obtain an interpretation of the Einstein tensor as equivalent to the energy-momentum tensor of gravitational fields (labelled by $^{[G]}$), in order to be able to quantize the gravitational field itself in a natural manner.

Let us start considering the Einstein field equations in the physical space-time V^4, in presence of external fields:

$$R^{<4>}_{\mu\nu} - \tfrac{1}{2}R^{<4>}g_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa T^{[F]}_{\mu\nu}. \tag{9.12}$$

And let us introduce the notation:

$$\kappa T^{[G]}_{\mu\nu} = -R^{<4>}_{\mu\nu} + \tfrac{1}{2} R^{<4>} g_{\mu\nu} + \Lambda g_{\mu\nu}. \tag{9.13}$$

We can interpret in a natural way $T^{[G]}_{\mu\nu}$ as the energy-momentum tensor of the gravitational field and write now the Einstein equations as an energetic balance between the gravitational and non-gravitational fields:

$$T^{[G]}_{\mu\nu} + T^{[F]}_{\mu\nu} = 0, \tag{9.14}$$

instead of embedding gravity within the geometry of space-time.

In particular from the calculations developed in chapter 8 we are able to evaluate $T^{[G]}_{\mu\nu}$ in correspondence to the Robertson-Walker metric.

In fact we have:

$$\kappa T^{[G]}_{00} = -\frac{3\dot{a}(t)^2}{c^2 a(t)^2} + \Lambda |c_g|^2 \equiv -\tfrac{3}{105}\Lambda |c_g|^2 + \Lambda |c_g|^2 \equiv \tfrac{34}{35}\Lambda |c_g|^2, \tag{9.15}$$

$$\begin{aligned} \kappa T^{[G]}_{11} &= -\left[-\frac{2\ddot{a}(t)}{c^2 a(t)} - \frac{\dot{a}(t)^2}{c^2 a(t)^2} + \Lambda |c_g|^2 \right] a(t)^2 \equiv \\ &\equiv -\Big(-\tfrac{3}{105}\Lambda |c_g|^2 + \Lambda |c_g|^2 \Big) a(t)^2 \equiv -\tfrac{34}{35}\Lambda |c_g|^2 a(t)^2, \end{aligned} \tag{9.16}$$

$$\begin{aligned} \kappa T^{[G]}_{22} &= -\left[-\frac{2\ddot{a}(t)}{c^2 a(t)} - \frac{\dot{a}(t)^2}{c^2 a(t)^2} + \Lambda |c_g|^2 \right] a(t)^2 r^2 \equiv \\ &\equiv -\Big(-\tfrac{3}{105}\Lambda |c_g|^2 + \Lambda |c_g|^2 \Big) a(t)^2 r^2 \equiv -\tfrac{34}{35}\Lambda |c_g|^2 a(t)^2 r^2, \end{aligned} \tag{9.17}$$

$$\begin{aligned} \kappa T^{[G]}_{33} &= -\left[-\frac{2\ddot{a}(t)}{c^2 a(t)} - \frac{\dot{a}(t)^2}{c^2 a(t)^2} + \Lambda |c_g|^2 \right] a(t)^2 r^2 \sin^2\theta \equiv \\ &\equiv -\Big(-\tfrac{3}{105}\Lambda |c_g|^2 + \Lambda |c_g|^2 \Big) a(t)^2 \equiv -\tfrac{34}{35}\Lambda |c_g|^2 a(t)^2 r^2 \sin^2\theta. \end{aligned} \tag{9.18}$$

In order to consider what happens in presence of more gravitons we introduce a coefficient c_{pg} for each graviton and we choose c_g .

We define:

$$|c_g|^2 = \sum_p |c_{gp}|^2. \tag{9.19}$$

Then the total energy-momentum tensor for the gravitational field will result:

$$\begin{aligned}
\kappa T_{00}^{[G]} &= \tfrac{34}{35}\sum_p \Lambda |c_{gp}|^2, \\
\kappa T_{11}^{[G]} &= -\tfrac{34}{35}\sum_p \Lambda |c_{gp}|^2\, a(t)^2, \\
\kappa T_{22}^{[G]} &= -\tfrac{34}{35}\sum_p \Lambda |c_{gp}|^2\, a(t)^2 r^2, \\
\kappa T_{33}^{[G]} &= -\sum_p \Lambda |c_{gp}|^2 a(t)^2 r^2 \sin^2\theta.
\end{aligned} \tag{9.20}$$

Now we can identify:

$$\kappa\, \varrho^{[G]} c^2 = \tfrac{34}{35}\Lambda, \qquad \kappa\, \wp^{[G]} = -\tfrac{34}{35}\Lambda, \tag{9.21}$$

as the energy and pressure densities of the gravitational field as observed in V^4, which include both visible and dark contributions, pressure being here negative as a consequence of gravitational attraction. Of course the energy and pressure densities of the gravitational field are equal and opposite in sign respect to the mass-energy and pressure densities $\varrho, \wp$ given by (8.36), (8.37) arising from the non-gravitational fields, so that the balance of gravity and non-gravitational fields is exactly zero.

9.4 Quantization of the Gravitational Field

Let us now examine the Hamiltonian density of the gravitational field, which is given by the $T_{00}^{[G]}$ component of the energy-momentum tensor we have just evaluated:

$$\mathcal{H}^{[G]} = \tfrac{34}{35}\tfrac{1}{\kappa}\textstyle\sum_p \Lambda |c_{gp}|^2. \qquad (9.22)$$

Integrating on a space region $\mathcal{D}$ of volume V we get the total Hamiltonian of the gravitational field enclosed within this region:

$$H^{[G]} = \int_{\mathcal{D}} \mathcal{H}^{[G]} \sqrt{|g|}\,\mathrm{d}^3 x = \tfrac{34}{35}\tfrac{V}{\kappa}\textstyle\sum_p \Lambda |c_{gp}|^2. \qquad (9.23)$$

Note that if $\mathcal{D}$ is assumed to be the whole universe at instant t the volume becomes time dependent.

Now we introduce the frequencies $\omega_{[g]p}$ through the relations:

$$\sqrt{\tfrac{34}{35}\tfrac{V}{\kappa}\Lambda}\; c_{gp} = \sqrt{\hbar\omega_{[g]p}}\; \mathsf{a}_{[g]p}. \qquad (9.24)$$

The square modulus yields:

$$\tfrac{34}{35}\tfrac{V}{\kappa}\Lambda |c_{gp}|^2 = \tfrac{1}{2}\hbar\omega_{[g]p}\left(\mathsf{a}^*_{[g]p}\mathsf{a}_{[g]p} + \mathsf{a}_{[g]p}\mathsf{a}^*_{[g]p}\right). \qquad (9.25)$$

Then the Hamiltonian becomes:

$$H^{[G]} = \tfrac{1}{2}\textstyle\sum_p \hbar\omega_{[g]p}\left(\mathsf{a}^*_{[g]p}\mathsf{a}_{[g]p} + \mathsf{a}_{[g]p}\mathsf{a}^*_{[g]p}\right). \qquad (9.26)$$

Quantization results by replacing the coefficients $\mathsf{a}^*_{[g]p}, \mathsf{a}_{[g]p}$ with the quantum creation and annihilation operators $\mathsf{a}^+_{[g]p}, \boldsymbol{a}_{[g]p}$, by the correspondence rules:

$$\mathsf{a}^*_{[g]p} \longrightarrow \boldsymbol{a}^+_{[g]p} \qquad \mathsf{a}_{[g]p} \longrightarrow \boldsymbol{a}_{[g]p}. \qquad (9.27)$$

The coefficient $c_{[g]p}$ being arbitrary it can always be adjusted in such a way to fit the right commutation relations for the operators $\mathsf{a}^+_{[g]p}, \mathsf{a}_{[g]p}$:

$$\boldsymbol{a}_{[g]p}\boldsymbol{a}^+_{[g]p} - \boldsymbol{a}^+_{[g]p}\boldsymbol{a}_{[g]p} = \boldsymbol{I}, \qquad (9.28)$$

thanks to which we obtain:

$$H^{[G]} = \sum_p \hbar\omega_{[g]p}\Big(a^{+}_{[g]p}a_{[g]p} + \tfrac{1}{2}\Big), \tag{9.29}$$

which provides also for the gravitational field a quantized Hamiltonian in the usual form known in *q.e.d.*

Conclusion

At the end of this book we may simply conclude that we have attempted to explore some intriguing problems on field unification and particles theory, with cosmological implications.

A first step was exploited in the *part I* of the book where we have proposed a unification of the concepts of particles and waves, based on the mathematical formal identity of the Hamilton-Jacobi equations governing the wave front motion and the mechanics of a family of particles. The relevant consequence of such approach is that unification implies that all motions, both of waves and particles, take place at the speed of light c. This circumstance seems to prepare, in some way, the conceptual unification also of relativity and quantum mechanics. In fact, according to this scheme, interactions can be included into the metric of space-time, rather than by a additional potentials which would break the identity of the Hamiltonians of waves and particles. Not surprisingly, as it happens in the *standard model*, when some tight symmetry condition is required, particles loose a rest mass. Here the strong condition imposing to waves and particles to travel all at the speed of light destroys their rest mass, unless an higher dimensional space-time embedding the experimentally known physical environment is supposed.

Furthermore, as we have seen in *part II* of the book, the assumption of an extended space-time with more than four dimensions provides the opportunity of unifying all the known fundamental interaction fields (*bosons*) within the metric tensor, when the dimension number is raised up to 16. So the eigenvectors of the metric tensor can be interpreted as vector potentials of the fundamental interactions. When a suitable gauge choice in the *extra* space-time V^{16} – which does not affect gauge invariance within the physical sub-space-time V^4 – is assumed, the dependence of the fields on the *extra* co-ordinates $x^{\underline{i}}$ may be assumed to affect only the scalar gauge function Φ (which is not an observable), and a non-vanishing rest mass of the particles arises, which is related to the mass of the scalar boson governed by Φ. This approach leads to a mechanism to generate particle mass which is different from the usual Higgs one, even if it is always based on the existence of a scalar boson.

The *fermion* fields governing matter particles (*leptons, quarks*) resulted automatically included in *extra* components $a_{\underline{l}}^{(\bar{\sigma})}$ of the metric tensor eigenvectors.

Remarkably, introducing new auxiliary variables $\lambda_{(\bar{\sigma})\bar{\mu}}$, the field equations could be written by means of an equivalent system in which also Maxwellian equations appear beside the Einstein equations.

In the last chapters we have examined possible applications of the theory to cosmology and to quantization of the gravitational field.

More is of course to be developed, *e.g.,* about universe anisotropies related to *dark matter/dark energy*, charge quantization, co-ordinate compactification, possible extended string-like solutions, etc. In any case the two approaches to wave-particle unification and field unification we have proposed has appeared to us worthy of being at least partially explored.

Appendices

Appendix A

Non-Linear Wave Propagation (Non Covariant Theory)

In order to facilitate the reader who is unfamiliar with the topic, here we present some notes on the elements of non-linear wave propagation theory we have employed in chapter 2.

A.1 Wave Kinematics

Let us consider a regular function $\varphi(t, x^{\bar{i}})$. We can interpret the equation:

$$\varphi(t, x^{\bar{i}}) = 0, \qquad \bar{i} = 1, 2, \cdots, n-1, \tag{A.1}$$

as the Cartesian equation of a wave-front traveling across space.

Physically it is supposed that something happens to a physical field, along this wave-front, as we will specify in the next §A.1.2.

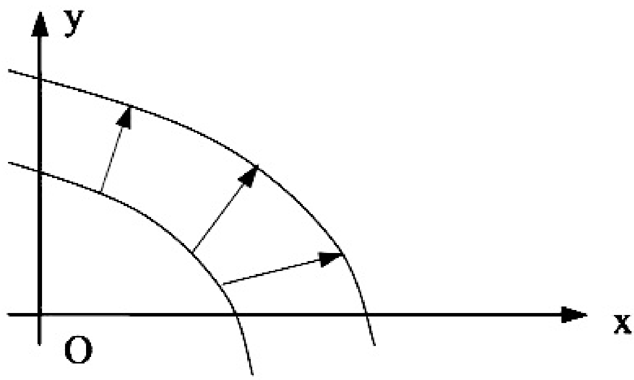

Figure A.1. *A traveling wave-front in two dimensions.*

Let us now consider some point on the wave-front, related to the initial condition $x_0^{\bar{i}}$ and actual position given by the function $x^{\bar{i}}(t)$.

On a geometrical stand-point the equations:

$$x^{\bar{i}} \equiv x^{\bar{i}}(t), \qquad \bar{i} = 1, 2, \cdots, n-1, \tag{A.2}$$

are the parametric equations of the trajectory of some point of the wave-front, considered as endowed with its own individuality. During motion of this point we have:

$$\varphi\big(t, x^{\bar{i}}(t)\big) = 0. \tag{A.3}$$

Differentiation respect to time of (A.3) yields:

$$\varphi_{,t} + \varphi_{,\bar{i}} \frac{\mathrm{d}x_{\bar{i}}}{\mathrm{d}t} = 0, \tag{A.4}$$

where:

$$\varphi_{,t} = \frac{\partial \varphi}{\partial t}, \qquad \varphi_{,\bar{i}} = \frac{\partial \varphi}{\partial x^{\bar{i}}}. \tag{A.5}$$

A.1.1 Ray and Normal Velocity

The trajectories described by each point of the wave-front are called *rays* and the vector of components:

$$V^{\bar{i}} = \frac{\mathrm{d}x^{\bar{i}}}{\mathrm{d}t}, \tag{A.6}$$

is referred to as the *ray velocity* of the wave in the considered point.

The vector of components $\varphi_{,\bar{i}}$, being the gradient of the function φ respect to the space co-ordinates, is locally orthogonal to the wave-front manifold at time t. So it results helpful to introduce also the unit vector:

$$n_{\bar{i}} = \frac{\varphi_{,\bar{i}}}{|\nabla\varphi|}, \qquad |\nabla\varphi| = \sqrt{-g^{\bar{i}\bar{k}}\varphi_{\bar{i}}\varphi_{\bar{k}}}, \tag{A.7}$$

which is defined at any point for which $|\nabla\varphi| \neq 0$.

Now, from (A.4) we have soon:

$$\frac{\varphi_{,t}}{|\nabla\varphi|} + V^{\bar{i}} n_{\bar{i}} = 0. \tag{A.8}$$

Introducing the notation:

$$\lambda = -\frac{\varphi_{,t}}{|\nabla\varphi|}, \tag{A.9}$$

we have from (A.8):

$$\lambda = V^{\bar{i}} n_{\bar{i}}. \tag{A.10}$$

Then it results immediate do interpret λ as the *normal speed* of the wave-front in each of its points. As we will see λ plays an important role in non-linear wave theory, since it represents the rate of the wave displacement across the space.

A.1.2 A Classification of Waves from an Analytic View-Point

A relevant classification of non-linear waves concerns the analytical properties of the field function $\boldsymbol{u}$ the waves are solutions to. We distinguish:

1. *Simple waves,* when the field $\boldsymbol{u}$ is a regular function of φ, *i.e.*:

$$\boldsymbol{u} \equiv \boldsymbol{u}(\varphi), \tag{A.11}$$

   and it is assumed that:

$$\varphi = x^{\bar{i}} n_{\bar{i}} - \lambda(\varphi) t. \tag{A.12}$$

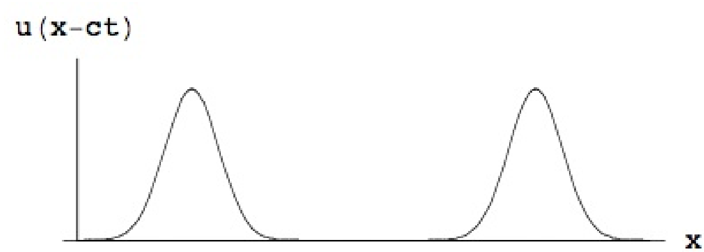

Figure A.2. *A simple wave traveling along the x axis with normal speed $\lambda = c$.*

   This kind of regular waves represents a generalization to non-linear wave propagation theory of the *plane waves* known in linear theory.

2. *Discontinuity waves* when the directional derivative of the field $\boldsymbol{u}$, along $n_{\bar{i}}$ (*normal derivative*) is discontinuous across the wave-front (*weak discontinuity*).

3. *Shock waves* when the field $\boldsymbol{u}$ itself is discontinuous across the wave-front (*strong discontinuity*).

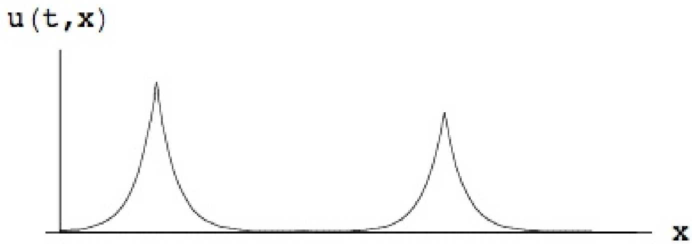

Figure A.3. *A discontinuity wave traveling along the x axis.*

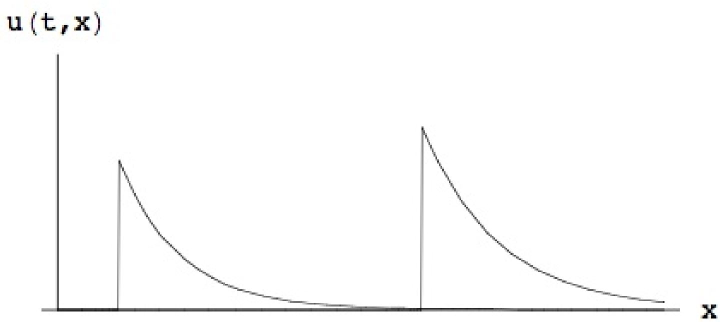

Figure A.4. *A shock wave traveling along the x axis.*

A.2 Wave Dynamics

Until now we have considered some elements of the *kinematics* describing wave motion, *i.e.,* the evolution of the wave-front across the space.

Now we want to say something about the *wave dynamics* concerning the field equations governing the filed $\boldsymbol{u}$, the waves are solutions to.

A.2.1 Qausi-Linear Systems

We start observing that the most general system of N non-linear first order partial differential equations, for the field unknowns $\boldsymbol{u} \equiv (u_J)$, $J = 1, 2, \cdots, N$, assumes the following form:

$$F_I\left(\frac{\partial u_J}{\partial t}, \frac{\partial u_J}{\partial x^{\bar{i}}}, u_J, t, x^{\bar{i}}\right) = 0, \qquad I, J = 1, 2, \cdots, N. \tag{A.13}$$

We remember that any system of higher order equations may be always reduced to a system of first order equations, provided that new variables and equations are added to the system.

When F_I is a differentiable function, as it generally happens for physical fields and (A.13) holds at the initial time (say $t = t_0$), the previous system results to be equivalent to a new one which is linear respect to all first order derivatives.

In fact, differentiating (A.13) respect to t, we obtain:

$$\frac{\partial F_I}{\partial v_J}\frac{\partial v_J}{\partial t} + \frac{\partial F_I}{\partial w_{J\bar{i}}}\frac{\partial w_{J\bar{i}}}{\partial t} + \frac{\partial F_I}{\partial u_J}\frac{\partial u_J}{\partial t} + \frac{\partial F_I}{\partial t} = 0, \tag{A.14}$$

where we have introduced the notations:

$$v_J = \frac{\partial u_J}{\partial t}, \qquad w_{Ji} = \frac{\partial u_J}{\partial x^i}. \tag{A.15}$$

Now, thanks to Schwarz condition, we can write the new equivalent system as:

$$\frac{\partial F_I}{\partial v_J}\frac{\partial v_J}{\partial t} + \frac{\partial F_I}{\partial w_{J\bar{i}}}\frac{\partial v_J}{\partial x^{\bar{i}}} = -\frac{\partial F_I}{\partial u_J}v_J - \frac{\partial F_I}{\partial t}, \tag{A.16}$$

$$\frac{\partial u_J}{\partial t} = v_J, \tag{A.17}$$

$$\frac{\partial w_{J\bar{i}}}{\partial t} - \frac{\partial v_J}{\partial x^{\bar{i}}} = 0, \tag{A.18}$$

$$F|_{t=t_0} = 0. \tag{A.19}$$

Since F_I does not involve the derivatives of v_J, w_{Ji}, the system results to be linear respect to such derivatives.

A system of partial differential equations involving higher order derivatives of the field variables at most linearly is said to be a *quasi-linear* system. Its general form is given by:

$$\mathsf{A}^0 \boldsymbol{u}_{,t} + \mathsf{A}^{\bar{i}} \boldsymbol{u}_{,\bar{i}} = \boldsymbol{f}, \qquad \boldsymbol{u}_{,t} = \frac{\partial \boldsymbol{u}}{\partial t}, \qquad \boldsymbol{u}_{,\bar{i}} = \frac{\partial \boldsymbol{u}}{\partial x^{\bar{i}}}. \tag{A.20}$$

The matrices $\mathsf{A}^0, \mathsf{A}^{\bar{i}}$ and the vector $\boldsymbol{f}$, may be functions of the field $\boldsymbol{u} \equiv \boldsymbol{u}(t, x^{\bar{i}})$ and $t, x^{\bar{i}}$.

A.2.2 Systems of Balance Laws

A physically relevant special case of quasi-linear systems is offered by the *systems of balance laws.* Such systems enjoy the property that the coefficient matrices assume the form of gradients respect to the field variables. Now when:

$$\mathsf{A}^0 = \frac{\partial \boldsymbol{f}^0}{\partial \boldsymbol{u}}, \qquad \mathsf{A}^{\bar{i}} = \frac{\partial \boldsymbol{f}^{\bar{i}}}{\partial \boldsymbol{u}}, \tag{A.21}$$

$\boldsymbol{f}^0, \boldsymbol{f}^{\bar{i}}$ being differentiable functions of the field $\boldsymbol{u}$ and possibly of $t, x^{\bar{i}}$, the system of field equations assumes the form of a set of balance laws:

$$\boldsymbol{f}^0_{\ ,t} + \boldsymbol{f}^{\bar{i}}_{\ ,\bar{i}} = \boldsymbol{f}. \tag{A.22}$$

A.2.3 Lagrangian Systems

The Lagrangian systems represent a class of very relevant systems of balance laws, in physical field theories. If we consider some field ϕ (which may be a scalar, or a vector or a tensor of any rank), which is governed by a Lagrangian density $\mathcal{L}(\phi, \phi_{,t}, \phi_{,\bar{i}})$, yielding the Euler-Lagrange equations:

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \phi_{,t}}\right) + \frac{\partial}{\partial x^{\bar{i}}}\left(\frac{\partial \mathcal{L}}{\partial \phi_{,\bar{i}}}\right) = \frac{\partial \mathcal{L}}{\partial \phi}, \tag{A.23}$$

we can reduce (A.23) to a system of first order equations:

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial v}\right) + \frac{\partial}{\partial x^{\bar{i}}}\left(\frac{\partial \mathcal{L}}{\partial w_{\bar{i}}}\right) = \frac{\partial \mathcal{L}}{\partial \phi}, \tag{A.24}$$

$$\frac{\partial w_{\bar{i}}}{\partial t} - \frac{\partial v}{\partial x^{\bar{i}}} = 0, \tag{A.25}$$

$$\frac{\partial \phi}{\partial t} = v, \tag{A.26}$$

which assumes also the compact form (A.22) if we set:

$$\boldsymbol{f}^0 = \begin{pmatrix} \frac{\partial \mathcal{L}}{\partial v} \\ \\ w_{\bar{j}} \\ \\ \phi \end{pmatrix}, \qquad \boldsymbol{f}^{\bar{i}} = \begin{pmatrix} \frac{\partial \mathcal{L}}{\partial w_i} \\ \\ -\delta_{ij} v \\ \\ 0 \end{pmatrix}, \qquad \boldsymbol{f} = \begin{pmatrix} \frac{\partial \mathcal{L}}{\partial \phi} \\ \\ 0 \\ \\ v \end{pmatrix}. \tag{A.27}$$

We emphasize that when the Lagrangian density depends only on the derivatives of the field ϕ the system assumes the simpler form of a set of *conservations laws* involving the only field variables v, $w_{\bar{j}}$.

In the latter case the last condition is unnecessary and we have simply:

$$\boldsymbol{f}^0 = \begin{pmatrix} \frac{\partial \mathcal{L}}{\partial v} \\ \\ w_{\bar{j}} \end{pmatrix}, \qquad \boldsymbol{f}^{\bar{i}} = \begin{pmatrix} \frac{\partial \mathcal{L}}{\partial w_{\bar{i}}} \\ \\ -\delta_{ij} v \end{pmatrix}, \qquad \boldsymbol{f} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}. \tag{A.28}$$

A.2.4 Hyperbolic Systems

In order that the waves related to some field variable travel with real and finite normal speeds, so that a field theory may be consistent with relativity, it is required that the system of field equations is *hyperbolic.*

A quasi-linear system (A.20) is said to be *hyperbolic* if and only if the two following conditions are fulfilled:

1. The matrix A^0 is non-singular, *i.e.*:

$$\det \mathsf{A}^0 \neq 0, \tag{A.29}$$

2. The generalized eigenvalue problem:

$$\left(\mathsf{A}_n - \lambda \mathsf{A}^0\right) \boldsymbol{d} = 0, \qquad \mathsf{A}_n = \mathsf{A}^{\bar{i}} n_{\bar{i}}, \tag{A.30}$$

allows only *real* eigenvalues and a *basis* of eigenvectors, for any choice of the unit vector of components $n_{\bar{i}}$.

A.2.5 Simple Waves

Here we limit ourselves to consider only the essential features of the *simple waves,* which are enough for the purposes of the present book, therefore we will not deal with neither *discontinuity waves* nor *shock waves.*

Let us consider a hyperbolic system of quasi-linear equations with vanishing production term $\boldsymbol{f}$:

$$\mathsf{A}^0 \boldsymbol{u}_{,t} + \mathsf{A}^{\bar{i}} \boldsymbol{u}_{,\bar{i}} = 0. \tag{A.31}$$

It follows that the system (A.31) is fulfilled by a class of *regular solutions* which exhibit the form of *simple waves, i.e.*:

$$u \equiv u(\varphi), \qquad \varphi = x_n - \lambda(\varphi)t, \qquad x_n = x^{\bar{i}} n_{\bar{i}}. \tag{A.32}$$

Note that the definition of φ, in the non-linear case, becomes an implicit definition, since λ is, in its turn, a function of φ, being dependent on $\boldsymbol{u}(\varphi)$, as the matrix of which it is an eigenvalue.

In fact, by direct calculation we have:

$$\boldsymbol{u}_{,t} = \boldsymbol{u}_{,\varphi}\varphi_{,t}, \qquad \boldsymbol{u}_{,\bar{i}} = \boldsymbol{u}_{,\varphi}\varphi_{,\bar{i}}, \qquad \boldsymbol{u}_{,\varphi} = \frac{\partial \boldsymbol{u}}{\partial \varphi}. \tag{A.33}$$

Now the derivatives of $\varphi = x_n - \lambda(\varphi)t$ result to be:

$$\varphi_{,t} = -\frac{\lambda}{1 + \lambda_{,\varphi} t}, \qquad \varphi_{,\bar{i}} = \frac{n_{\bar{i}}}{1 + \lambda_{,\varphi} t}. \tag{A.34}$$

Substitution into (A.31) leads to:

$$(\mathsf{A}_n - \lambda \mathsf{A}_0)\, \boldsymbol{u}_{,\varphi} = 0. \tag{A.35}$$

We conclude that:

1. Simple waves propagate with normal speeds λ given by the eigenvalues of the problem (A.35) and their amplitudes have normal derivatives $\boldsymbol{u}_{,\varphi}$ which are eigenvectors of the same problem.
2. Hyperbolicity of the system of field equations ensures that their normal speeds are real finite quantities.
3. Finally we observe that it is possible to provide an operational rule to obtain directly the eigenvalue problem (A.35), starting from the system of differential equations (A.31).

The latter rule is given by the correspondence:

$$\frac{\partial}{\partial_t} \longrightarrow -\lambda\frac{\partial}{\partial\varphi}, \qquad\qquad \frac{\partial}{\partial x^{\bar{i}}} \longrightarrow n_i\frac{\partial}{\partial\varphi}, \tag{A.36}$$

avoiding to evaluate explicitly the coefficient matrices in order to obtain their eigenvalues and eigenvectors.

Appendix B

Non-Linear Wave Propagation (Covariant Theory)

In this appendix we present an outline of the elements of non-linear wave propagation theory in an explicitly covariant formulation.

B.1 Wave Kinematics

Let us consider a regular function $\varphi(x^{\bar{\alpha}})$. We can interpret the equation:

$$\varphi(x^{\bar{\alpha}}) = 0, \qquad \bar{\alpha} = 0, 1, 2, \cdots, n-1, \tag{B.1}$$

as the Cartesian equation of a wave-sheet living in space-time, which may be thought of as the world sheet swept by a wave-front as some evolutionary parameter σ increases its value. Physically it is supposed that something happens to some physical field on this wave-front, as we will specify in the next §B.2. Let us now consider some point on the wave-front, related to an actual position given by the function $x^{\bar{\alpha}}(\sigma)$ with $x^{\bar{\alpha}}(0)$, when $\sigma = 0$. On a geometrical stand-point the equations:

$$x^{\bar{\alpha}} \equiv x^{\bar{\alpha}}(\sigma), \qquad \bar{\alpha} = 0, 1, 2, \cdots, n-1, \tag{B.2}$$

are the parametric equations of the path of some point of the wave-front, considered as endowed with its own individuality. During motion we have:

$$\varphi(x^{\bar{\alpha}}(\sigma)) = 0. \tag{B.3}$$

When the wave-sheet is time-like the parameter σ can be chosen as equal to the *proper time* multiplied by c (*proper length*) along the trajectory.

While light-like paths do not allow a similar choice, the proper time being null. So σ cannot be chosen in such a way that it is a V^n invariant scalar.

Differentiation respect to σ of (B.3) yields:

$$\varphi_{,\bar{\alpha}} \frac{\mathrm{d}x^{\bar{\alpha}}}{\mathrm{d}\sigma} = 0, \tag{B.4}$$

where:

$$\varphi_{,\bar{\alpha}} = \frac{\partial \varphi}{\partial x^{\bar{\alpha}}}. \tag{B.5}$$

B.1.1 Ray Velocity

The vector of components:

$$V^{\bar{\alpha}} = \frac{\mathrm{d}x^{\bar{\alpha}}}{\mathrm{d}\sigma}, \tag{B.6}$$

is referred to as the *ray velocity* of the wave in the considered point.

The vector of components $\varphi_{,\bar{\alpha}}$, being the gradient of the function φ respect to the space co-ordinates, is orthogonal to the wave-sheet manifold at the point $x^{\bar{\alpha}}(\sigma)$. So it results helpful to introduce also the unit vector:

$$n_{\bar{\alpha}} = \frac{\varphi_{\bar{\alpha}}}{\sqrt{\left| g^{\bar{\alpha}\bar{\beta}} \varphi_{\bar{\alpha}} \varphi_{\bar{\beta}} \right|}}, \tag{B.7}$$

which is defined at any point for which $g^{\bar{\alpha}\bar{\beta}} \varphi_{\bar{\alpha}} \varphi_{\bar{\beta}} \neq 0$.

Now, from (B.4) we have soon:

$$V^{\bar{\alpha}} \varphi_{,\bar{\alpha}} = 0. \tag{B.8}$$

And:

$$V^{\bar{\alpha}} n_{\bar{\alpha}} = 0. \tag{B.9}$$

Manifestly the normal velocity becomes identically vanishing when evaluated by contraction with $n_{\bar{\alpha}}$. So its introduction becomes useless.

If desired it may be defined anyway, even within a covariant formulation of the theory, respect to any time direction $\tau^{\bar{\alpha}}$ which is different from $V^{\bar{\alpha}}$, thanks to the decomposition (covariant Lorentz transformation):

$$V^{\bar{\alpha}} = \frac{\tau^{\bar{\alpha}} - \frac{\lambda}{c}\nu^{\bar{\alpha}}}{\sqrt{1 - \frac{\lambda^2}{c^2}}}, \qquad n_{\bar{\alpha}} = \frac{\nu_{\bar{\alpha}} - \frac{\lambda}{c}\tau_{\bar{\alpha}}}{\sqrt{1 - \frac{\lambda^2}{c^2}}}, \tag{B.10}$$

where:

$$\tau^{\bar{\alpha}}\tau_{\bar{\alpha}} = 1, \qquad \nu^{\bar{\alpha}}\nu_{\bar{\alpha}} = -1, \qquad \tau^{\bar{\alpha}}\nu_{\bar{\alpha}} = 0. \tag{B.11}$$

It follows:

$$\lambda = c\,\frac{V^{\bar{\alpha}}\nu_{\bar{\alpha}}}{\sqrt{1 + (V^{\bar{\alpha}}\nu_{\bar{\alpha}})^2}}, \tag{B.12}$$

and also:

$$\lambda = -c\,\frac{\tau^{\bar{\alpha}}\,n_{\bar{\alpha}}}{\sqrt{1 + (\tau^{\bar{\alpha}}\,n_{\bar{\alpha}})^2}}. \tag{B.13}$$

B.1.2 A Classification of Waves from an Analytic View-Point

A relevant classification of non-linear waves concerns the analytical properties of the field function $\boldsymbol{u}$ the waves are solutions to. We distinguish:

1. *Simple waves* when the field $\boldsymbol{u}$ is a regular function of φ, *i.e.*:

$$\boldsymbol{u} \equiv \boldsymbol{u}(\varphi), \tag{B.14}$$

and it is assumed that:

$$\varphi = V_{\bar{\alpha}}\,x^{\bar{\alpha}}. \tag{B.15}$$

This kind of regular waves represents a generalization to non-linear wave propagation theory of the *plane waves* known in linear theory.

2. *Discontinuity waves* when the directional derivative of the field $\boldsymbol{u}$, along $\varphi_{,\bar{\alpha}}$ (*normal derivative*) is discontinuous across the wave sheet (*weak discontinuity*).

3. *Shock waves* when the field $\boldsymbol{u}$ itself is discontinuous across the wave front (*strong discontinuity*).

B.2 Wave Dynamics

Until now we have considered some elements of the *kinematics* describing wave motion, *i.e.,* the evolution of the wave-sheet across the space. Here we want to say something about the *wave dynamics* concerning the field equations governing the filed $\boldsymbol{u}$, the waves are solutions to.

B.2.1 Qausi-Linear Systems

A system of partial differential equations involving higher order derivatives of the field variables at most linearly is said to be a *quasi-linear* system. Its more general covariant form is given by:

$$\mathsf{A}^{\bar{\alpha}}\boldsymbol{u}_{,\bar{\alpha}} = \boldsymbol{f}, \qquad \boldsymbol{u}_{,\bar{\alpha}} = \frac{\partial \boldsymbol{u}}{\partial x_{\bar{\alpha}}}. \tag{B.16}$$

The matrices $\mathsf{A}^{\bar{\alpha}}$ and the vector $\boldsymbol{f}$, may be functions of the field $\boldsymbol{u} \equiv \boldsymbol{u}(x^{\bar{\alpha}})$ and possibly $x^{\bar{\alpha}}$.

We point out that the partial derivatives may always be replaced by covariant derivatives, if required, since the additional contributions arising from the connection coefficients do not involve the derivatives of the components of the field $\boldsymbol{u}$, but only the field itself. Therefore they simply contribute to modify the production term $\boldsymbol{f}$. In that case the quasi-linear system assumes the form:

$$A^{\bar{\alpha}}\boldsymbol{u}_{;\bar{\alpha}} = \widehat{\boldsymbol{f}}, \qquad \widehat{\boldsymbol{f}} = \boldsymbol{f} + \Delta\boldsymbol{f}(\boldsymbol{u}), \qquad \Delta\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{u}_{;\bar{\alpha}} - \boldsymbol{u}_{,\bar{\alpha}}. \tag{B.17}$$

B.2.2 Systems of Balance Laws

A physically most relevant special case of quasi-linear systems is offered by the *systems of balance laws.* Such systems enjoy the property that the coefficient matrices assume the form of gradients respect to the field variables, *i.e.*:

$$A^{\bar{\alpha}} = \frac{\partial \boldsymbol{f}^{\bar{\alpha}}}{\partial \boldsymbol{u}}, \tag{B.18}$$

$\boldsymbol{f}^{\bar{\alpha}}$ being differentiable functions of the field $\boldsymbol{u}$ and possibly of $x^{\bar{\alpha}}$. Then the system of field equations assumes the compact form:

$$\boldsymbol{f}^{\bar{\alpha}}_{\ ,\bar{\alpha}} = \boldsymbol{f}. \tag{B.19}$$

When the covariant derivatives are required it becomes:

$$\boldsymbol{f}^{\bar{\alpha}}_{\ ;\bar{\alpha}} = \widehat{\boldsymbol{f}}. \tag{B.20}$$

B.2.3 Lagrangian Systems

The Lagrangian systems represent a class of most relevant systems of balance laws, in physical field theories.

If we consider some field ϕ (which may be a scalar, or a vector or a tensor of any rank), which is governed by a Lagrangian density $\mathcal{L}(\phi, \phi_{,\alpha})$, yielding the Euler-Lagrange equations:

$$\left(\frac{\partial \mathcal{L}}{\partial \phi_{,\bar{\alpha}}}\right)_{,\bar{\alpha}} = \frac{\partial \mathcal{L}}{\partial \phi}, \tag{B.21}$$

we can reduce (B.21) to a system of first order equations:

$$\left(\frac{\partial \mathcal{L}}{\partial w_{\bar{\alpha}}}\right)_{,\bar{\alpha}} = \frac{\partial \mathcal{L}}{\partial \phi}, \tag{B.22}$$

$$w_{\bar{\alpha},\bar{\beta}} - w_{\bar{\beta},\bar{\alpha}} = 0, \tag{B.23}$$

which assumes also the compact form (B.19) if we set:

$$\boldsymbol{f}^{\bar{\alpha}} = \begin{pmatrix} \frac{\partial \mathcal{L}}{\partial w_{\bar{\alpha}}} \\ \\ g_{\bar{\alpha}\bar{\gamma}}\, w_{\bar{\beta}} - g_{\bar{\alpha}\bar{\beta}}\, w_{\bar{\gamma}} \end{pmatrix}, \qquad \boldsymbol{f} = \begin{pmatrix} \frac{\partial \mathcal{L}}{\partial \phi} \\ \\ 0 \end{pmatrix}. \tag{B.24}$$

When the Lagrangian density depends only on the derivatives of the field ϕ the system assumes the form of a *system of conservations laws* ($\boldsymbol{f} = 0$).

B.2.4 Hyperbolic Systems

In order that the waves related to some field variable travel with real and finite normal speeds, so that a field theory is consistent with relativity, it is required that the system of field equations is *hyperbolic.*

A quasi-linear system (B.16) is said to be *hyperbolic* along the time direction defined by a time-like congruence $\tau_{\bar{\alpha}}$ if and only if the two following conditions are fulfilled:

1. The matrix $\mathsf{A}^{\bar{\alpha}}\,\tau_{\bar{\alpha}}$ is not singular, *i.e.*:

$$\det\left(\mathsf{A}^{\bar{\alpha}}\,\tau_{\bar{\alpha}}\right) \neq 0, \tag{B.25}$$

2. The generalized eigenvalue problem:

$$\left(\nu_{\bar{\alpha}} - \tfrac{\lambda}{c}\tau_{\bar{\alpha}}\right)\mathsf{A}^{\bar{\alpha}}\boldsymbol{d} = 0, \tag{B.26}$$

allows only *real* eigenvalues and a *basis* of eigenvectors, for any choice of the unit vector of components $n_{\bar{\alpha}}$.

We observe that, thanks to (B.7) and (B.10), eq. (B.26) is equivalent to the more compact forms:

$$\varphi_{,\bar{\alpha}}\,\mathsf{A}^{\bar{\alpha}}\boldsymbol{d} = 0, \tag{B.27}$$

or:

$$n_{\bar{\alpha}}\,\mathsf{A}^{\bar{\alpha}}\boldsymbol{d} = 0. \tag{B.28}$$

B.2.5 Simple Waves

Let us consider a hyperbolic system of quasi-linear equations with vanishing production term $\boldsymbol{f}$:

$$\mathsf{A}^{\bar{\alpha}}\boldsymbol{u}_{,\bar{\alpha}} = 0. \tag{B.29}$$

It follows that the system (B.29) is fulfilled by a class of *regular solutions* which exhibit the form of *simple waves, i.e.*:

$$\boldsymbol{u} \equiv \boldsymbol{u}(\varphi), \qquad \varphi = n_{\bar{\alpha}}\,x^{\bar{\alpha}}. \tag{B.30}$$

In fact, by direct computation we have:

$$\boldsymbol{u}_{,\bar{\alpha}} = \boldsymbol{u}_{,\varphi}\varphi_{,\bar{\alpha}}, \qquad \boldsymbol{u}_{,\varphi} = \frac{\partial \boldsymbol{u}}{\partial \varphi}. \tag{B.31}$$

Now the derivatives of $\varphi = n_{\bar{\alpha}}\,x^{\bar{\alpha}}$ result to be:

$$\varphi_{,\bar{\alpha}} = \frac{n_{\bar{\alpha}}}{1 + n_{\bar{\beta},\varphi}\,x^{\bar{\beta}}}, \tag{B.32}$$

since n_α is in its turn a function of φ. Substitution into (B.29) leads to:

$$\varphi_{,\bar{\alpha}} \boldsymbol{A}^{\bar{\alpha}} \boldsymbol{u}_{,\varphi} = 0. \tag{B.33}$$

We conclude that:

1. Simple waves propagate with normal speeds λ given by the eigenvalues of the problem (B.27) and their amplitudes have normal derivatives $\boldsymbol{u}_{,\varphi}$ which are eigenvectors of the same problem.

2. Hyperbolicity of the system of field equations ensures that their normal speeds are real finite quantities.

3. Finally we observe that it is possible to provide an operational rule to obtain directly the eigenvalue problem (B.27), starting from the system of differential equations (B.29). The latter rule is given by the correspondence:
$$\frac{\partial}{\partial x^{\bar{\alpha}}} \quad \longrightarrow \quad \varphi_{,\bar{\alpha}} \frac{\partial}{\partial \varphi}, \tag{B.34}$$
avoiding to evaluate explicitly the coefficient matrices in order to obtain their eigenvalues and eigenvectors.

Appendix C

Covariant Hamiltonian Mechanics

In this appendix we present some notes on covariant Lagrangian and Hamiltonian formulation of relativistic particle dynamics which can include also the massless particle case.

C.1 Covariant Lagrangian of a Particle

Let S be a scalar action integral governing the dynamics of a relativistic particle:

$$S = \int_{\sigma_1}^{\sigma_2} L \, \mathrm{d}\sigma, \tag{C.1}$$

where:

$$L = \tfrac{1}{2} \mathcal{K} g_{\bar{\mu}\bar{\nu}} \, \dot{x}^{\bar{\mu}} \, \dot{x}^{\bar{\nu}}, \tag{C.2}$$

dot denoting, here, the derivative respect to the parameter σ, which maps the position $x^{\bar{\alpha}}(\sigma)$ of some particle in space-time and $\mathcal{K}$ being an arbitrary dimensional constant.

1. When the particle rest mass m is non-vanishing, one can choose σ equal to the particle *proper time* multiplied by the speed of light c (*proper length*) and $\mathcal{K}$ equal to mc^2 so that L has the physical dimension of an energy and it is a covariant scalar.

2. In the limiting case when the particle has zero rest mass, σ cannot be identified with proper time which is null, and cannot be a V^n scalar parameter. For instance it can be chosen equal to the proper time in V^4 if this latter is non-vanishing. In this case it results invariant under

general transformations of the co-ordinates x^0, x^1, x^2, x^3, being a V^4 scalar.

Or it may be defined by the local time co-ordinate $x^0 = ct$. In any case the time co-ordinate x^0 can be always re-scaled in such a way that it results $g_{00} = 1$ in any co-ordinate frame. So σ will result to be a scalar respect to any co-ordinate change preserving the condition $g_{00} = 1$, leaving arbitrary the remaining components of the metric tensor.

The Lagrange motion equation of the particle:

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\frac{\partial L}{\partial \dot{x}^{\bar{\alpha}}} - \frac{\partial L}{\partial x^{\bar{\alpha}}} = 0, \tag{C.3}$$

being:

$$\frac{\partial L}{\partial \dot{x}^{\bar{\alpha}}} = \mathcal{K}\dot{x}_{\bar{\alpha}}, \qquad \frac{\partial L}{\partial x^{\bar{\alpha}}} = \tfrac{1}{2}\mathcal{K} g_{\bar{\mu}\bar{\nu}\,,\bar{\alpha}}\; \dot{x}^{\bar{\mu}}\,\dot{x}^{\bar{\nu}}, \tag{C.4}$$

assumes the explicit form:

$$\ddot{x}_{\bar{\alpha}} - \tfrac{1}{2} g_{\bar{\mu}\bar{\nu}\,,\bar{\alpha}}\,\dot{x}^{\bar{\mu}}\,\dot{x}^{\bar{\nu}} = 0. \tag{C.5}$$

The latter equation is equivalent to the geodesic condition:

$$\frac{\mathrm{d}\dot{x}_{\bar{\alpha}}}{\mathrm{d}\sigma} - \Gamma^{\bar{\mu}}_{\bar{\alpha}\bar{\nu}}\,\dot{x}_{\bar{\mu}}\,\dot{x}^{\bar{\nu}} = 0, \tag{C.6}$$

being:

$$\Gamma^{\bar{\mu}}_{\bar{\alpha}\bar{\nu}} = \tfrac{1}{2} g^{\bar{\mu}\bar{\rho}}\left(g_{\bar{\alpha}\bar{\rho},\bar{\nu}} + g_{\bar{\nu}\bar{\rho},\bar{\alpha}} - g_{\bar{\alpha}\bar{\nu}\,,\bar{\rho}}\right). \tag{C.7}$$

C.2 Covariant Hamiltonian

The Hamiltonian is the Legendre transform of the Lagrangian, *i.e.*:

$$H = p_{\bar{\alpha}}\,\dot{x}^{\bar{\alpha}} - L, \tag{C.8}$$

with the conjugate canonical momentum:

$$p_{\bar{\alpha}} = \frac{\partial \mathcal{L}}{\partial \dot{x}^{\bar{\alpha}}}. \tag{C.9}$$

It follows:

$$H = \frac{1}{2\mathcal{K}} g^{\bar{\mu}\bar{\nu}} p_{\bar{\mu}} p_{\bar{\nu}}, \tag{C.10}$$

resulting:

$$\dot{x}_{\bar{\alpha}} = \frac{1}{\mathcal{K}} p_{\bar{\alpha}}. \tag{C.11}$$

This formulation has the advantage of being explicitly covariant when $m \neq 0$ and to hold, in any case also if $m = 0$.

When the particle has non-vanishing rest mass, H is a covariant scalar.

Then the canonical Hamilton equations:

$$\dot{p}_{\bar{\alpha}} = -\frac{\partial H}{\partial x^{\bar{\alpha}}}, \qquad \dot{x}^{\bar{\alpha}} = \frac{\partial H}{\partial p_{\bar{\alpha}}}, \tag{C.12}$$

become:

$$\dot{p}_{\bar{\alpha}} = -\frac{1}{2\mathcal{K}} g^{\bar{\mu}\bar{\nu}}{}_{,\bar{\alpha}}\, p_{\bar{\mu}} p_{\bar{\nu}}, \qquad \dot{x}^{\bar{\alpha}} = \frac{1}{\mathcal{K}} g^{\bar{\alpha}\bar{\mu}} p_{\bar{\mu}}. \tag{C.13}$$

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Short Introduction to the Book

This book sketches in detail, in its first part, some possible intriguing suggestions in order to conceive, and deal within a unified scheme, the dual aspects of particle and wave coexisting and conflicting in theoretical physics from the beginning of the 20^{th} century, so opening a door to a deep harmonization of Relativity and Quantum Mechanics. And in the second part it presents a new approach to field unification, including gravity, together with the standard model of elementary particles and scalar boson, within the frame of multidimensional general relativistic universe. Cosmological application of the proposed theory involves non-trivial predictions about the still unexplained universe flatness, dark matter and dark energy. A way to gravity quantization appears also practicable.

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